1. Consider a star of radius R where the temperature T and mean particle mass μ are uniform inside, except for a tiny layer on the surface in which the temperature drops from T to a much lower value.

   (a) Derive an expression for the period of the fundamental radial oscillation of this star.

   Solution: This is simpler than the uniform-density star; the speed of sound is uniform in the interior:

   $$v_s = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{\gamma \rho k T}{\rho \mu}} = \sqrt{\frac{\gamma k T}{\mu}}$$

   so

   $$\Pi = 4\int_0^R \frac{dr}{v_s} = 4\sqrt{\frac{\mu}{\gamma k T}}\int_0^R dr = 4R\sqrt{\frac{\mu}{\gamma k T}}$$

   (b) Suppose a star like this were observed to have a radius $R = 1.5R_\odot$ and we could tell from its spectrum that it has the same mean particle mass and specific-heat ratio as the Sun. Suppose furthermore it oscillates with a period of 2 hours. What is its interior temperature T?

   Solution: Such a star would be similar in its properties to a δ Scuti star:

   $$T = \frac{\mu}{\gamma k}\left(\frac{4R}{\Pi}\right)^2 = 2.44 \times 10^7 \text{ K}$$

2. Clarinets can play more than one note, even without pushing on the keys. Presumably stars can too.

   (a) What is the next-lowest pitch (frequency) that can be played by the clarinet considered in the class notes? This mode is called the **first overtone**; the lowest-frequency mode is called the **fundamental**. Give your answer in Hz and/or in musical notation.

   Solution: There has to be a pressure antinode (a maximum) at the closed end (the mouthpiece) and a pressure node at the open end. If the clarinet bore is of length L the wavelength of the fundamental mode will be $\lambda_1 = 4L$ since $L = \lambda/4$ wavelengths will obey the boundary conditions. The next mode that would fit places one additional pressure node within the clarinet, so that $\lambda_2 = 4L/3$ since $L = 3\lambda/4$. This is the first overtone.

   Thus, the frequency is a factor of three higher than the fundamental:

   $$f_1 = v_s/\lambda_1 = v_s/4L = 142 \text{ Hz}$$
   $$f_2 = v_s/\lambda_2 = 3v_s/4L = 3f_1 = 426 \text{ Hz}$$

   The fundamental is near D below middle C. The first overtone is near A in the middle of the staff. (Experts will note this clarinet is tuned a little flat.)

   (b) What is the next-shortest period of oscillation (i.e., the first overtone) of the uniform-density star considered in class?

   Solution: Much like the clarinet, the frequency of the star's first overtone is $3\times$ higher than its fundamental, so the period is three times shorter:

   $$\Pi_2 = \frac{4}{3}\int_0^R \frac{dr}{v_s} = \frac{1}{3}\sqrt{\frac{6\pi}{\gamma G \rho}} = \sqrt{\frac{2\pi}{3\gamma G\rho}}$$

(c) The brightest classical Cepheid variable in the sky is Polaris, the North Star (α Ursae Minoris). Its pulsation period is 3.97 days and its amplitude is 0.03 mag. Its spectrum and color show that it has an effective temperature of 7200 K. It has a couple of companion stars from which its mass can be determined to be $4.3M_\odot$ and its distance has been measured with trigonometric parallax, yielding from its flux a luminosity of $2200L_\odot$. Estimate the periods of Polaris' fundamental and first-overtone pulsations. In which mode is Polaris likely to be pulsating?

Solution: We need the average density in order to calculate oscillation periods:

$$L = 4\pi R^2 \sigma T_e^4$$

$$\rho = \frac{3M}{4\pi R^2} = \frac{3\sigma T_e^4 M}{L}$$

$$\Pi_1 = \sqrt{\frac{6\pi}{\gamma G\rho}} = \sqrt{\frac{2\pi L}{\gamma\sigma T_e^4 GM}} = 8.7\times 10^5 \text{ s} = 10 \text{ days}$$

$$\Pi_2 = \frac{1}{3}\Pi_1 = 2.9\times 10^5 \text{ s} = 3.3 \text{ days}$$

The assumption of constant density is crude and we know this should introduce an error, but all the same it looks like the estimate of the period of Polaris is a better match for the first overtone than for the fundamental mode. Note that detailed models also give the same answer.

Oscillation in the first overtone is rare among Cepheids. That Polaris is doing so is thought to have something to do with its peculiarly small oscillation amplitude; perhaps Polaris is in the process of switching its mode of pulsation from the first overtone to the fundamental.

Note that D.G. Turner et al. (2013) made a determination of the distance to Polaris based on careful measurements of the mean brightness and spectral type compared to other stars of the same spectral type and accurately known distances and found a new distance 30% smaller than that measured by *Hipparcos* using trig parallax. The measurement by Turner reduces the luminosity to the point that the fundamental mode agrees better with the observed pulsation period than the first overtone. Turner et al. have been extremely careful, but the *Hipparcos* parallax measurement had high signal-to-noise. Most astronomers will bet on *Hipparcos* and the overtone pulsation and give large odds because many more things can go wrong in the treatment by Turner et al. than with *Hipparcos* in this distance range.

3. **Brown Dwarfs**: Consider a star of such very low mass as to be only marginally capable of thermonuclear heat production. Under the assumptions that the star is all hydrogen ($Z = A = 1$), that gravity is balanced by *nonrelativistic electron degeneracy pressure*, and that protons, at the same temperature and pressure as the electrons, *act as an ideal gas*, derive the equation relating the star's central temperature T_c to its total mass M. If $T_c \geq 3\times 10^6$ K is required to sustain the pp chain fusion reactions, what is the minimum mass of a luminous star? Express your answer in solar masses ($M_\odot$) and compare it to the mass of Jupiter ($1\ M_{\rm Jup} = 2\times 10^{30}$ g).

"Stars" with mass less than this minimum never undergo hydrogen fusion energy generation. These are the brown dwarfs.

Solution: The central pressure and mass density in a body supported by electron degeneracy pressure are given by

$$P_c = 0.77 \frac{GM^2}{R^4}$$
$$\rho_c = 1.43 \frac{M}{R^3}$$

Since $Z = A = 1$ by assumption, the protons have the same pressure and temperature as the electrons and will behave like an ideal gas. Thus, we have an additional expression for P_c:

$$P_c = \frac{\rho_c k T_c}{m_p}$$

and therefore

$$T_c = \frac{m_p P_c}{k \rho_c} = \frac{0.77 G M m_p}{1.43 k R}$$

Using the mass-radius relation derived in class (with $Z/A = 1$),

$$R = 0.114 \frac{h^2}{G m_e m_p^{5/3}} M^{-1/3}$$

we can eliminate R from the expression for T_c:

$$\begin{aligned} T_c &= \frac{0.77 G M m_p}{1.43 k} \frac{G m_e m_p^{5/3}}{0.114 h^2} M^{1/3} \\ &= 4.72 \frac{G^2 m_e m_p^{8/3}}{h^2 k} M^{4/3} \\ &= 1.24 \times 10^{-36} M^{4/3} \text{ K g}^{-4/3} \end{aligned}$$

We know that T_c needs to be greater than about 3×10^6 K to ignite the pp chain of fusion reactions. Therefore, from this expression the star must have a mass of at least

$$M = 0.8 \times 10^{36} \text{ g} \, (T/\text{K})^{3/4} = 6 \times 10^{31} \text{ g} = 0.03 M_\odot$$

to undergo fusion (and thus to be luminous in the long term). This limit corresponds to a body about 30 times more massive than Jupiter ($0.001 M_\odot$) and about 30 times less massive than the Sun. This result is close to the present accepted value of $0.08 M_\odot$ produced by much more detailed calculations; we did not get the exact answer because we ignored the Coulomb interaction between electrons, which is more involved. The important point is that stars cannot be made with arbitrarily small masses because electron degeneracy pressure keeps small objects from getting hot enough for fusion.

4. Begin with the relativistic form of electron degeneracy pressure and the expressions for central pressure from weight and central density in a relativistic-degenerate equation of state:

$$P_e = 0.123 \; hc n_e^{4/3} \qquad P_c = 11 \frac{GM^2}{R^4} \qquad \rho_c = 12.9 \frac{M}{R^3}$$

Substitute $n_e = Z\rho / A m_p$ and manipulate to obtain both an expression for the electron degeneracy pressure in a star made of material with nuclear charge Z and mass number A, and an expression for the (Stoner-Andersen-Chandrasekhar) maximum mass of such a star. You will thus fill in the steps left out in arriving at the results in the lecture on white dwarfs.

Calculate the maximum mass of a carbon white dwarf, expressing your answer in solar masses.

Solution: We substitute $n_e = Z\rho/Am_p$ into $P_e = 0.123hcn_e^{4/3}$ and obtain

$$P_e = 0.123\frac{hc}{m_p^{4/3}}\left(\frac{Z}{A}\right)^{4/3}\rho^{4/3}$$

The SAC mass is that for which the relativistic electron pressure balances the pressure due to the weight, i.e.,

$$11\frac{GM_{\mathrm{SAC}}^2}{R^4} = 0.123\frac{hc}{m_p^{4/3}}\left(\frac{Z}{A}\right)^{4/3}\left(12.9\frac{M_{\mathrm{SAC}}}{R^3}\right)^{4/3}$$

Canceling the factors of R gives

$$M_{\mathrm{SAC}}^{2/3} = 0.340\frac{hc}{Gm_p^2}\left(\frac{Z}{A}\right)^{4/3}m_p^{2/3}$$

or

$$\begin{aligned} M_{\mathrm{SAC}} &= 0.198\left(\frac{hc}{Gm_p^2}\right)^{3/2}\left(\frac{Z}{A}\right)^2 m_p \\ &= 1.78\times 10^{57}m_p \\ &= 2.88\times 10^{33}\ \mathrm{g} \\ &= 1.44M_\odot \end{aligned}$$

for $Z/A = 0.5$, as is the case for carbon.

5. Two oboe players can hear each other; one hovers just outside a black hole's horizon at $r = 1.01R_{\mathrm{sch}}$ and the other is at rest far away from the black hole. (This is the George Lucas version of outer space.) Each plays an $A4$ note ($f = 440$ Hz). What is the frequency of the note each hears played by the other?

Solution: The frequency $f = 440$ Hz corresponds to a period $1/f = 2.27$ ms. If that is the note played by the oboist near the black hole such that $\Delta\tau = 1/f$ then the distant observer hears

$$\Delta t = \frac{\Delta\tau}{\sqrt{1-2GM/rc^2}} = \frac{2.27\ \mathrm{ms}}{\sqrt{1-1/1.01}} = 22.8\ \mathrm{ms} \qquad f = \frac{1}{\Delta t} = 43.9\ \mathrm{Hz}$$

This is an F more than two octaves below middle C and barely within the range of human hearing.

If it is the note played by the "inertial" oboist ($\Delta t = 1/f$) then the oboist by the black hole hears

$$\Delta\tau = \Delta t\sqrt{1-\frac{2GM}{rc^2}} = 2.27\ \mathrm{ms}\sqrt{1-\frac{1}{1.01}} = 0.226\ \mathrm{ms} \qquad f = \frac{1}{\Delta\tau} = 4.42\ \mathrm{kHz}$$

This is a C# more than four octaves above middle C, but still in the human range of hearing.