1. **Stable black hole orbits**:
   (a) Derive an expression for the orbital frequency of the innermost stable circular orbit (ISCO) around a black hole $f_{\rm ISCO}$ in terms of the black hole's mass.
   (b) The black hole in GRO J1655-40 is observed to have a mass of $M = 6.0M_\odot$ and emits X-rays which exhibit quasiperiodic oscillations at a frequency of 450 Hz. It is thought that this frequency indicates material in orbit. If the black hole were not spinning, how far (radially, in cm) from the ISCO would this orbit be, in the view of a distant observer?
   (c) If the black hole were not spinning, how far (radially, in cm) would this orbit be from the ISCO in the view of an observer in the vicinity of these orbits? Presume the local observer to be capable of measuring the distance instantaneously.

2. **Black hole evaporation**: Hawking radiation from a black hole with mass M is emitted at a rate and spectrum identical to a blackbody with temperature $T = hc^3/16\pi^2 kGM$.
   (a) Calculate the effective temperature and luminosity of a very small black hole with mass $M = 10^{15}$ g. At what wavelength is the peak luminosity? In which part of the electromagnetic spectrum is this wavelength?
   (b) Show that Hawking radiation leads to a *decrease* in the mass of black holes at a rate

$$\frac{dM}{dt} = -\frac{A}{c^2 M^2}$$

   where A is a constant. Compute the value of A. Then use this expression to derive a formula for the lifetime of a black hole with initial mass M_0.
   (c) Calculate the mass of a black hole with a lifetime of 1.4×10^{10} yr. As we will see, the Universe is about this old; thus any remaining primordial black holes created in the Big Bang must be heavier than the mass you calculate, assuming they have not accreted more material.

3. The pressure and temperature at the center of the Sun are $P_c = 2.1 \times 10^{17}$ dyne cm^{-2} and $T_c = 15.7 \times 10^6$ K. What are the central pressure and temperature of a $1.5M_\odot$ star with the same composition as the Sun?

   (*Hint*: Empirical measurements of eclipsing binaries show that stellar mass and radius are close to linearly proportional with coefficient 1 in solar units. Use this to infer the missing piece of information needed to solve this problem.)

4. **A helium star**: Consider stars with composition of a type called extreme "Population II": objects made only of hydrogen (fraction X by mass) and helium (fraction $1 - X$ by mass), with negligible quantities of everything else.
   (a) Starting with the definition $\mu = \frac{\rho}{n}$ for the average particle mass, where ρ is the mass density and n is the number of particles (electrons, protons, and helium nuclei) per unit volume, show that the average mass in an extreme Population II star is

$$\mu = \frac{4m_p}{5X + 3}$$

   Hint: Recall that each hydrogen atom contributes two particles when ionized, and each helium atom contributes three.

(b) Show that the luminosity in a star that is supported by ideal gas pressure scales with average particle mass, total mass, and internal temperature as

$$L \propto \mu^7 M^5 T^{0.5}$$

(c) A $1M_\odot$ Population II star with $X = 0.75$ is quite similar to the Sun; it has luminosity $L_\odot$ and radius $R_\odot$. Use this information and the result of part b to estimate the luminosity, radius, and surface temperature of a $1M_\odot$ pure helium star ($X = 0$).

5. **Radiation pressure and hydrostatic equilibrium in giant stars**: Recall that the momentum of a photon is $p = \frac{E}{c}$.

   (a) Derive an expression for the radiation pressure, defined as the outward momentum per unit time per unit area, delivered to the outer layers of a star with luminosity L, assuming that all the photons from the interior are absorbed in those layers.

   (b) Calculate the total radiation force on the outer layers of a star with $L = L_\odot$. How does the radially outward force from radiation compare to the force of gravity on the layers if the layers lie at $R = 100R_\odot$, have mass $m = 10^{-6}M_\odot$, and the rest of the star has a mass $M = 1M_\odot$?

   (c) Repeat part b for a typical AGB star: $L = 7000L_\odot$, $T_e = 3000$ K, same masses. (*Hint*: You will need to first work out the star's radius.)

6. **Time above the horizon**

   (a) Plot the amount of time (in hours) spent above the horizon each day as a function of declination (in degrees) for observations made at latitudes of +43° (Rochester) and +31° (Kitt Peak, Arizona).

   (b) Suppose that you want to go observing in Rochester "tonight" (March 4), and take pictures of M1 (the Crab Nebula) and M8 (the Lagoon Nebula). Look up the coordinates of these objects, and the sidereal time at midnight "tonight," and estimate when each of these objects rises and sets. Is this a good time of year to observe M1 and M8? (Note: You must show your work, and few-minute accuracy will suffice.)