1. **Stable black hole orbits**:

   (a) Derive an expression for the orbital frequency of the innermost stable circular orbit (ISCO) around a black hole $f_{\rm ISCO}$ in terms of the black hole's mass.

   Solution: The ISCO has a coordinate radius of $r_{\rm ISCO} = 3R_{\rm Sch} = 6GM/c^2 = 53.2$ km. The circumference of circular orbits and the orbital velocity are the same in all reference frames, so

$$f = \frac{1}{P} = \frac{v_\phi}{2\pi r} = \frac{\sqrt{GM/r}}{2\pi r} = \frac{1}{2\pi}\sqrt{\frac{GM}{r^3}}$$

$$f_{\rm ISCO} = \frac{1}{2\pi}\sqrt{\frac{GM}{r_{\rm ISCO}^3}} = \frac{1}{2\pi}\sqrt{GM\left(\frac{c^2}{6GM}\right)^3} = \frac{c^3}{12\pi G\sqrt{6}}\frac{1}{M}$$

   (b) The black hole in GRO J1655-40 is observed to have a mass of $M = 6.0M_\odot$ and emits X-rays which exhibit quasiperiodic oscillations at a frequency of 450 Hz. It is thought that this frequency indicates material in orbit. If the black hole were not spinning, how far (radially, in cm) from the ISCO would this orbit be, in the view of a distant observer?

   Solution: Invert the previous result for f to estimate the coordinate radius r of the orbit corresponding to 450 Hz:

$$r = \left(\frac{GM}{4\pi^2 f^2}\right)^{1/3} = 4.64\times10^6 \text{ cm} = 46.4 \text{ km}$$

   Therefore, the radial distance from the ISCO orbit is

$$\Delta r = 3R_{\rm Sch} - r = \frac{6GM}{c^2} - \left(\frac{GM}{4\pi^2 f^2}\right)^{1/3} = 6.8\times10^5 \text{ cm} = 6.8 \text{ km}$$

   (c) If the black hole were not spinning, how far (radially, in cm) would this orbit be from the ISCO in the view of an observer in the vicinity of these orbits? Presume the local observer to be capable of measuring the distance instantaneously.

   Solution: Inside the gravitational well we have to account for curvature effects. Using the integral from Recitation 5 we have

$$\begin{aligned}\Delta\mathcal{L} &= \int_r^{r_{\rm ISCO}} \frac{dr'}{\sqrt{1 - R_{\rm Sch}/r'}} = R_{\rm Sch}\int_{r/R_{\rm Sch}}^{r_{\rm ISCO}/R_{\rm Sch}} \frac{du}{\sqrt{1-1/u}} \\ &= R_{\rm Sch}\left[\frac{1}{2}\ln\left(\frac{1/\sqrt{1-1/u}+1}{1/\sqrt{1-1/u}-1}\right) + \sqrt{u(u+1)}\right]_{r/R_{\rm Sch}}^{r_{\rm ISCO}/R_{\rm Sch}} \\ &= R_{\rm Sch}\left[\frac{1}{2}\ln\left(\frac{1+\sqrt{1-1/u}}{1-\sqrt{1-1/u}}\right) + \sqrt{u(u+1)}\right]_{r/R_{\rm Sch}}^{r_{\rm ISCO}/R_{\rm Sch}} \\ &= 8.4\times10^5 \text{ cm} = 8.4 \text{ km}\end{aligned}$$

2. **Black hole evaporation**: Hawking radiation from a black hole with mass M is emitted at a rate and

spectrum identical to a blackbody with temperature $T = hc^3/16\pi^2 kGM$.

(a) Calculate the effective temperature and luminosity of a very small black hole with mass $M = 10^{15}$ g. At what wavelength is the peak luminosity? In which part of the electromagnetic spectrum is this wavelength?

Solution: To a good approximation the BH radiates only electromagnetic energy as a spherical blackbody with circumference $2\pi R_{\text{Sch}}$, so

$$L = 4\pi R_{\text{Sch}}^2 \sigma T^4$$

Using $R_{\text{Sch}} = 2GM/c^2$, $T = hc^3/16\pi^2 kGM$, and $\sigma = (2\pi^5/15)(k^4/h^3c^2)$ we obtain

$$\begin{aligned} L &= 4\pi \left(\frac{2GM}{c^2}\right)^2 \left(\frac{2\pi^5}{15}\right) \left(\frac{k^4}{h^3c^2}\right) \left(\frac{hc^3}{16\pi^2 kGM}\right)^4 \\ &= \frac{32}{15(16)^4\pi^2} \frac{hc^6}{G^2M^2} \\ &= \frac{hc^6}{30720\pi^2 G^2 M^2} = \frac{A}{M^2} \end{aligned}$$

For $M = 10^{15}$ g the temperature and luminosity are

$$T = 1.23 \times 10^{11} \text{ K} \qquad L = \frac{hc^6}{30720\pi^2 G^2 M^2} = 3.6 \times 10^{15} \text{ erg/s}$$

According to Wien's Law, this corresponds to a blackbody that peaks at

$$\begin{aligned} \lambda_{\max} &= 0.29 \text{ cm K}/T = 2.4 \times 10^{-12} \text{ cm} \\ E &= hc/\lambda_{\max} = 9.93 \times 10^{-12} \text{ J} \\ &\approx 62 \text{ MeV} \end{aligned}$$

well into the γ-ray range.

(b) Show that Hawking radiation leads to a *decrease* in the mass of black holes at a rate

$$\frac{dM}{dt} = -\frac{A}{c^2M^2}$$

where A is a constant. Compute the value of A. Then use this expression to derive a formula for the lifetime of a black hole with initial mass M_0.

Solution: The rest energy of the black hole is $E = Mc^2$. As it radiates energy its rest energy decreases at an equal but opposite rate:

$$\begin{aligned} L &= -\frac{dE}{dt} \\ \frac{A}{M^2} &= -c^2\frac{dM}{dt} \\ \frac{dM}{dt} &= -\frac{A}{c^2M^2} \end{aligned}$$

where $A = hc^6/(30720\pi^2 G^2)$ was estimated in part (a). To find the lifetime we separate variables and integrate:

$$\int_0^\tau dt = -\frac{c^2}{A}\int_{M_0}^0 M^2\, dM$$
$$\tau = \frac{c^2}{A}\frac{M_0^3}{3} = \frac{10240\pi^2 G^2}{hc^4}M_0^3$$

(c) Calculate the mass of a black hole with a lifetime of 1.4×10^{10} yr. As we will see, the Universe is about this old; thus any remaining primordial black holes created in the Big Bang must be heavier than the mass you calculate, assuming they have not accreted more material.

Solution: From part (b),

$$M_0 = \left(\frac{hc^4\tau}{10240\pi^2 G^2}\right)^{1/3} = 1.74 \times 10^{14} \text{ g}$$

This is about 10% of the mass of a large mountain like Everest; astronomically, small asteroids a few hundred meters in diameter have similar masses.

3. The pressure and temperature at the center of the Sun are $P_c = 2.1 \times 10^{17}$ dyne cm^{-2} and $T_c = 15.7 \times 10^6$ K. What are the central pressure and temperature of a $1.5M_\odot$ star with the same composition as the Sun?

(*Hint*: Empirical measurements of eclipsing binaries show that stellar mass and radius are close to linearly proportional with coefficient 1 in solar units. Use this to infer the missing piece of information needed to solve this problem.)

Solution: According to the empirical mass-radius relation for the components of eclipsing binaries, mass and radius are close to linearly proportional with coefficient 1 in solar units. Therefore the radius of a $1.5M_\odot$ star is about $1.5R_\odot$. Using the scaling relations developed in class, we get

$$P_c = 2.1 \times 10^{17}(1.5)^2(1.5)^{-4} \text{ dyne cm}^{-2} = 9.3 \times 10^{16} \text{ dyne cm}^{-2}$$
$$T_c = 15.7 \times 10^6 (1.5)\left(\frac{1}{1.5}\right)(1) \text{ K} = 15.7 \times 10^6 \text{ K}$$

4. **A helium star**: Consider stars with composition of a type called extreme "Population II": objects made only of hydrogen (fraction X by mass) and helium (fraction $1 - X$ by mass), with negligible quantities of everything else.

(a) Starting with the definition $\mu = \frac{\rho}{n}$ for the average particle mass, where ρ is the mass density and n is the number of particles (electrons, protons, and helium nuclei) per unit volume, show that the average mass in an extreme Population II star is

$$\mu = \frac{4m_p}{5X + 3}$$

Hint: Recall that each hydrogen atom contributes two particles when ionized, and each helium atom contributes three.

Solution: We know that in terms of the number density of H and He, $\rho = \mu n = m_p n_{\rm H} + 4 m_p n_{\rm He}$ and $n = 2n_{\rm H} + 3n_{\rm He}$, so

$$\mu = \frac{\rho}{n} = \frac{n_{\rm H} + 4n_{\rm He}}{2n_{\rm H} + 3n_{\rm He}} m_p$$

We can eliminate $n_{\rm He}$ from this expression in favor of the hydrogen mass fraction X by noting that

$$m_p n_{\rm H} = X\rho \qquad\qquad 4m_p n_{\rm He} = (1 - X)\rho$$

Dividing the two expressions gives

$$n_{\rm He} = \frac{n_{\rm H}(1 - X)}{4X}$$

which we substitute back into the expression for μ:

$$\mu = \frac{n_{\rm H} + 4n_{\rm H}(1 - X)/4X}{2n_{\rm H} + 3n_{\rm H}(1 - X)/4X} m_p = \frac{4X + 4(1 - X)}{8X + 3(1 - X)} m_p = \frac{4m_p}{5X + 3}$$

(b) Show that the luminosity in a star that is supported by ideal gas pressure scales with average particle mass, total mass, and internal temperature as

$$L \propto \mu^7 M^5 T^{0.5}$$

Solution: Since the star is supported by ideal gas pressure,

$$P = nkT = \frac{\rho}{\mu} kT$$
$$T = \frac{P\mu}{\rho k} \propto \frac{GM^2 R^{-4}}{MR^{-3}} \frac{\mu}{k} \propto \frac{\mu M}{R}$$

The luminosity in the center of the star was given in class as

$$L \propto \frac{R^3 T^4}{R^2/\ell} \propto \ell R T^4$$

In low-mass stars $\ell \propto T^{3.5}\rho^{-2}$. Thus

$$L \propto \frac{RT^{7.5}}{\rho^2} \propto R\left(\frac{\mu M}{R}\right)^{7.5} \frac{R^6}{M^2} \propto \mu^{7.5} M^{5.5} R^{-0.5}$$

Using $T \propto \mu M/R$ one more time to eliminate R gives

$$L \propto \mu^7 M^5 T^{0.5}$$

(c) A $1M_\odot$ Population II star with $X = 0.75$ is quite similar to the Sun; it has luminosity $L_\odot$ and radius $R_\odot$. Use this information and the result of part b to estimate the luminosity, radius, and surface temperature of a $1M_\odot$ pure helium star ($X = 0$).

Solution: The standard $1M_\odot$ star has $X = 0.75$ so in its center the average particle mass is about $\mu = 0.59m_p$. In the He star with $X = 0$, $\mu = 4m_p/3$. Therefore, in terms of the normal star, the luminosity of the $1M_\odot$ He star is

$$\begin{aligned} L_{\rm He} &= \frac{\left(\mu^7 M^5 T^{0.5}\right)_{\rm He}}{\left(\mu^7 M^5 T^{0.5}\right)_{\rm X=0.75}} L(X=0.75) \\ &= \left(\frac{4m_p/3}{0.59m_p}\right)^7 \left(\frac{10^8\ \mathrm{K}}{1.57\times 10^7\ \mathrm{K}}\right)^{0.5} L_\odot \\ &= 760 L_\odot \end{aligned}$$

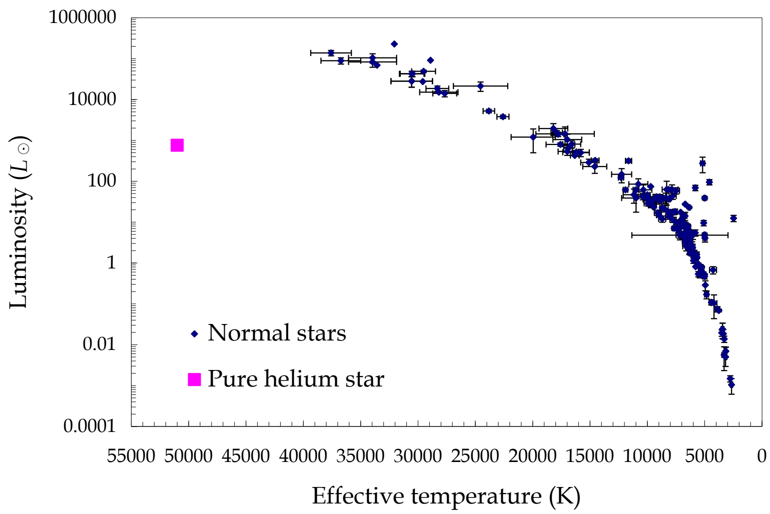
To get the radius we use $T \propto \mu M/R$ to express the radius in terms of the normal star:

$$\begin{aligned} R_{\rm He} &= R_{\rm H}\left(\frac{(\mu M/T)_{\rm He}}{(\mu M/T)_{\rm X=0.75}}\right) \\ &= R_\odot \frac{4m_p/3}{0.59m_p}\frac{1.57\times 10^7\ \mathrm{K}}{10^8\ \mathrm{K}} \\ &= 0.36 R_\odot \end{aligned}$$

The effective temperature is given by the usual expression $L = 4\pi R^2 \sigma T^4$, so

$$\begin{aligned} T_e &\propto \left(\frac{L}{R^2}\right)^{1/4} \\ T_e(\mathrm{He}) &= \left(\frac{L_{\rm He}}{R_{\rm He}^2}\frac{R_{\rm X=0.75}^2}{L_{\rm X=0.75}}\right)^{1/4} T_e(X=0.75) \\ &= \left(\frac{760}{0.36^2}\right)^{1/4} 5800\ \mathrm{K} \\ &= 51000\ \mathrm{K} \end{aligned}$$

As shown in the figure below, this places the He star well to the left of and below the main sequence, as determined using eclipsing binary stars.



5. **Radiation pressure and hydrostatic equilibrium in giant stars**: Recall that the momentum of a photon is $p = \frac{E}{c}$.

   (a) Derive an expression for the radiation pressure, defined as the outward momentum per unit time per unit area, delivered to the outer layers of a star with luminosity L, assuming that all the photons from the interior are absorbed in those layers.

   Solution: By outer layers we mean $r = R$:

   $$\begin{aligned} P_{\text{rad}} &= \frac{F}{A} \\ &= \frac{1}{4\pi R^2}\frac{dp}{dt} \\ &= \frac{1}{4\pi R^2}\frac{1}{c}\frac{dE}{dt} \\ \Aboxed{P_{\text{rad}} &= \frac{L}{4\pi R^2 c}} \end{aligned}$$

   (b) Calculate the total radiation force on the outer layers of a star with $L = L_\odot$. How does the radially outward force from radiation compare to the force of gravity on the layers if the layers lie at $R = 100R_\odot$, have mass $m = 10^{-6}M_\odot$, and the rest of the star has a mass $M = 1M_\odot$?

   Solution: $F = PA$ so

   $$F_{\text{rad}} = \frac{L}{c} = \frac{3.827 \times 10^{33}\ \text{erg/s}}{3 \times 10^{10}\ \text{cm/s}}$$

   $$\boxed{F_{\text{rad}} = 1.3 \times 10^{23}\ \text{dynes}}$$

   $$\begin{aligned} F_{\text{grav}} &= -\frac{GMm}{R^2} = -\frac{(6.674 \times 10^{-8}\ \text{dyn cm}^2\ \text{g}^{-2})(1.989 \times 10^{33}\ \text{g})^2(1 \times 10^{-}6)}{((100)(6.96 \times 10^{10}\ \text{cm}))^2} \\ &= -5.5 \times 10^{27}\ \text{dynes} \end{aligned}$$

   So even with the high luminosity radiation pressure is several orders of magnitude too small to hold up the star. Gas pressure holds up the star, and radiation pressure is too small to blow off the outer layers.

   (c) Repeat part b for a typical AGB star: $L = 7000L_\odot$, $T_e = 3000$ K, same masses. (*Hint*: You will need to first work out the star's radius.)

   Solution: First solve for the radius of the star:

   $$\begin{aligned} L &= 4\pi R^2 \sigma T_e^4 \\ R &= \sqrt{\frac{L}{4\pi\sigma T_e^4}} = \sqrt{\frac{(7000)(3.827 \times 10^{33}\ \text{erg/s})}{4\pi(5.6704 \times 10^{-5}\ \text{erg s}^{-1}\ \text{cm}^{-2}\ \text{K}^{-4})(3000\ \text{K})^4}} \\ R &= 2.2 \times 10^{13}\ \text{cm} = 310R_\odot \end{aligned}$$

Thus

$$F_{\rm rad} = \frac{L}{c} = \frac{(7000)(3.827\times10^{33}\ \text{erg/s})}{3\times10^{10}\ \text{cm/s}}$$
$$\boxed{F_{\rm rad} = 8.9\times10^{26}\ \text{dynes}}$$
$$F_{\rm grav} = -\frac{GMm}{R^2} = -\frac{(6.674\times10^{-8}\ \text{dyn cm}^2\ \text{g}^{-2})(1.989\times10^{33}\ \text{g})^2(1\times10^{-6})}{(2.2\times10^{13})^2}$$
$$F_{\rm grav} = -5.6\times10^{26}\ \text{dynes}$$

This time radiation wins out and the outer layers will be propelled further outward.

6. **Time above the horizon**

(a) Plot the amount of time (in hours) spent above the horizon each day as a function of declination (in degrees) for observations made at latitudes of +43° (Rochester) and +31° (Kitt Peak, Arizona).

Solution: First, get a formula for that time. For an object on the horizon, $ZA = 90°$, so $\cos ZA = 0$:

$$\begin{aligned}\cos ZA &= \cos HA_h \cos\delta\cos\lambda + \sin\delta\sin\lambda\\ 0 &= \\ HA_h &= \pm\cos^{-1}\left(-\frac{\sin\delta\sin\lambda}{\cos\delta\cos\lambda}\right)\end{aligned}$$

Note that $\cos^{-1}$ is double-valued, because $\cos x = \cos(-x)$. So the object rises when

$$HA_{h-} = -\cos^{-1}\left(-\frac{\sin\delta\sin\lambda}{\cos\delta\cos\lambda}\right)$$

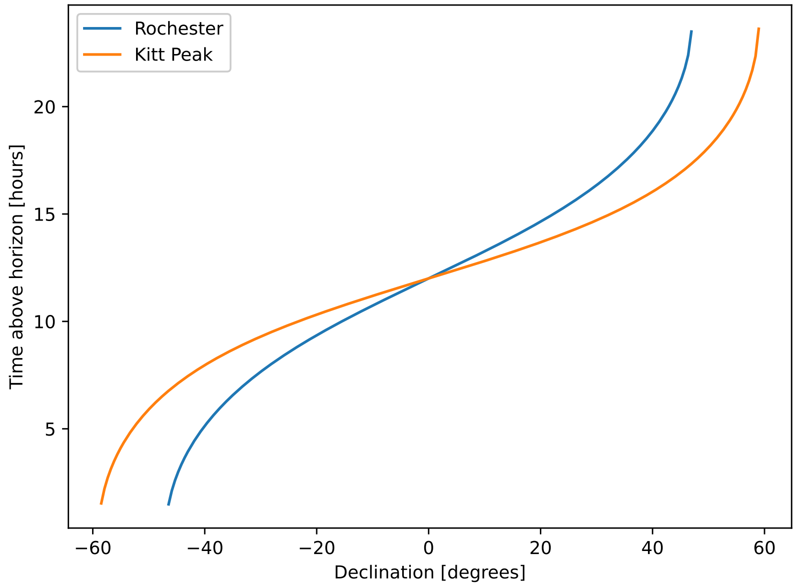
and sets when

$$HA_{h+} = +\cos^{-1}\left(-\frac{\sin\delta\sin\lambda}{\cos\delta\cos\lambda}\right)$$

taking HA hours to rise to the meridian and the same time to set. The total time in hours above the horizon is

$$\begin{aligned}\Delta t &= HA_{h+} - HA_{h-}\\ &= 2\cos^{-1}\left(-\frac{\sin\delta\sin\lambda}{\cos\delta\cos\lambda}\right)\end{aligned}$$

This is what we need to plot. We can put our latitudes and declinations in degrees, but note that we need an answer in hours, for which we multiply an answer in degrees by $\frac{24}{360}$. We also note that most plotting programs (including python) expect their angular inputs to be in radians, not degrees. The results (calculated in python) are shown below. Note that the curves cross at zero declination and twelve hours.



You will note while you plot that some of the declination values result in angles with cosines greater than 1 or less than -1:

$$-\frac{\sin\delta\sin\lambda}{\cos\delta\cos\lambda}\begin{cases}< -1 & \rightarrow \delta > \tan^{-1}(\cot\lambda)\\ > 1 & \rightarrow \delta < \tan^{-1}(-\cot\lambda)\end{cases}$$

From the shape of the rest of the curve, we can guess what this means. The first of these cases describes objects so far north in the sky that they never set; such objects are called *circumpolar*. The latter describes objects lying so far south that they never rise.

(b) Suppose that you want to go observing in Rochester "tonight" (March 4), and take pictures of M1 (the Crab Nebula) and M8 (the Lagoon Nebula). Look up the coordinates of these objects, and the sidereal time at midnight "tonight," and estimate when each of these objects rises and sets. Is this a good time of year to observe M1 and M8? (Note: You must show your work, and few-minute accuracy will suffice.)

Solution: The coordinates of these two objects are:

$$\alpha_{\rm M1} = 05^h34^m31.97^s = 5.58^h \qquad \delta_{\rm M1} = +22°00'52.1'' = 22.01°$$
$$\alpha_{\rm M8} = 18^h03^m18.06^s = 18.06^h \qquad \delta_{\rm M8} = -24°23.2' = -24.39°$$

From the sidereal time calculator that we used in recitation, we get

$$\mathrm{LST}_0 = 10^h43^m19^s = 10.72^h$$

for Rochester (longitude 77.6°W, latitude +43°). Thus, M1 reaches the meridian

$$5.58^h - 10.72^h = -5.14^h$$

before midnight, so within a few minutes of 6:51PM. M8 transits at

$$18.06^h - 10.72^h = 7.34^h$$

or 7:20AM. Now, according to

$$\Delta t = 2\cos^{-1}\left(-\frac{\sin\delta\sin\lambda}{\cos\delta\cos\lambda}\right)$$

M1 stays above Rochester's horizon for 14.95 hours, ±7.48 hours from the time that it transits. M8 stays up for 8.67 hours, ±4.33 hours from transit. Therefore, M1 rises at

$$-5.14^h - 7.48^h = -12.62^h = 11:22 \text{ AM}$$

and sets at

$$-5.14^h + 7.48^h = 2.33^h = 2:19 \text{ AM}$$

M8 rises at

$$7.34^h - 4.33^h = 3.01^h = 3:00 \text{ AM}$$

and sets at

$$7.34^h + 4.33^h = 11.67^h = 11:40 \text{ AM}$$

So we get about half a night for both M1 and M8, considering the times of sunset and sunrise (6:04PM, 6:39AM). It is a decent time of year to observe both of these objects.

You can check this with TheSkyX or Stellarium, both of which do a very precise calculation: indeed, all of our results come within a couple of minutes.