1. A certain star has a measured V magnitude equal to 13.54 and a measured B magnitude of 14.41. A U magnitude is measured that leads to a B-V color excess of $E(B-V) = 0.25$.

   (a) Calculate the *visual extinction* A_V toward the star, the star's *extinction-corrected V magnitude* and the *extinction-corrected B-V color index*, and the extinction-corrected apparent bolometric magnitude and effective temperature of the star. Bolometric corrections and effective temperatures can be found in Table 1.

(B-V)$_0$	T_e	BC	(B-V)$_0$	T_e	BC
-0.35	40000	-4.5	0.5	6320	-0.04
-0.31	31900	-3.34	0.53	6200	-0.05
-0.3	30000	-3.17	0.6	5920	-0.06
-0.26	24200	-2.5	0.64	5780	-0.07
-0.24	22100	-2.23	0.68	5610	-0.1
-0.2	18800	-1.77	0.72	5490	-0.15
-0.16	16400	-1.39	0.81	5240	-0.19
-0.14	15400	-1.21	0.92	4780	-0.25
-0.12	14500	-1.04	0.98	4590	-0.35
-0.09	13400	-0.85	1.15	4410	-0.65
-0.06	12400	-0.66	1.3	4160	-0.9
0	10800	-0.4	1.41	3920	-1.2
0.03	10200	-0.32	1.48	3680	-1.48
0.06	9730	-0.25	1.52	3500	-1.76
0.09	9260	-0.2	1.55	3360	-2.03
0.15	8620	-0.15	1.56	3230	-2.31
0.2	8190	-0.12	1.61	3120	-2.62
0.33	7240	-0.08	1.73	3050	-3.21
0.38	6930	-0.06	1.8	2940	-3.46
0.45	6540	-0.04	1.91	2640	-4.1
0.47	6450	-0.04			

Table 1: Color index, effective temperatures, and bolometric correction for main-sequence stars.

Solution: For dark diffuse clouds, recall from lecture that

$$R = \frac{A_V}{E(B-V)} = 3.06 \qquad \Longrightarrow A_V = 3.06E(B-V) = (3.06)(0.25) = 0.77$$

Therefore the corrected magnitude and color index are

$$V_0 = V - 0.77 = 12.78$$
$$(B-V)_0 = B - V - E(B-V) = 0.87 - 0.25 = 0.62$$

For this color index the bolometric correction is -0.065 and $T_e = 5850$ K. The apparent bolometric magnitude is $m = V_0 - 0.065 = 12.71$.

   (b) Suppose you had ignored extinction. Use the observed V magnitude and color index to infer a *bolometric magnitude* and *effective temperature* for the star, and compare your results to those of part a, assuming the star is main sequence. How large an error in luminosity is made by ignoring extinction?

Solution: Ignoring extinction we obtain B-V = 0.87, a bolometric correction of -0.22, a T_e = 5010 K and an apparent magnitude of $m = 13.32$. Therefore,

$$\Delta m = m_{\rm ext} - m_{\rm no\ ext} = 2.5 \log \left(\frac{f_{\rm no\ ext}}{f_{\rm ext}} \right) = 2.5 \log \left(\frac{L_{\rm no\ ext}}{L_{\rm ext}} \right)$$
$$\frac{L_{\rm no\ ext}}{L_{\rm ext}} = 10^{(m_{\rm ext} - m_{\rm no\ ext})/2.5} = 0.57$$

This is almost a factor of two error in luminosity.

(c) Estimate the *absolute bolometric magnitude* of the star, calculate its distance, and estimate its spectral type. You can use the data file `ZAMS.txt` on the course website.

Solution: With its corrected colors the star appears to be Solar type, so its class is about G2. Therefore, taking a value $M = 4.68$ for its absolute bolometric magnitude, its distance r is given by

$$m = M + 5 \log (r/10 \text{ pc})$$
$$r = (10 \text{ pc}) 10^{(m-M)/5} = (10 \text{ pc}) 10^{(12.71-4.68)/5}$$
$$r = 403.6 \text{ pc}$$

2. Suppose a newly-formed O5 star ($T_e = 35000$ K, $R = 18 R_\odot$) lies within a dusty shell of radius 0.2 pc. Under the assumption that the grains are small spherical blackbodies heated by light from the central star, calculate the temperature of the grains in the dusty shell. At what wavelength do the dust grains shine brightest?

Solution: The star emits a total power $L = 4\pi R^2 \sigma T_e^4$ and does so isotropically. At the radius r of the dust shell the stellar flux is $f = L/4\pi r^2$. If the spherical grains have radius a, their geometrical cross section is πa^2, so they absorb a power

$$P_{\rm abs} = fA = \frac{L}{4\pi r^2} \pi a^2$$

If the grain is treated as a spherical blackbody of surface area $4\pi a^2$, then it emits $L_{\rm gr} = 4\pi a^2 \sigma T_{\rm gr}^2$. If it is in thermal equilibrium with starlight then the input and output power (luminosity) are equal:

$$P_{\rm abs} = L_{\rm gr}$$
$$\frac{L}{4\pi r^2} \pi a^2 = \frac{R^2 \sigma T_e^4}{r^2} \pi a^2 = 4\pi a^2 \sigma T_{\rm gr}^4$$
$$T_{\rm gr} = T_e \left(\frac{R}{2r} \right)^{1/2} = (35000 \text{ K}) \left(\frac{(18)(7 \times 10^{10} \text{ cm})}{(2)(0.2)(3.1 \times 10^{18} \text{ cm})} \right)^{1/2}$$
$$T_{\rm gr} = 35.3 \text{ K}$$

By Wien's Law, the blackbody with this temperature has peak brightness

$$\lambda_{\rm max} = \frac{0.29 \text{ cm K}}{35.3 \text{ K}} = 82.2 \ \mu\text{m}$$

which lies in the far-IR range.

3. Suppose a spherical cloud is made of pure molecular hydrogen and has a uniform number density 10^6 cm^{-3}, uniform temperature 100 K, and mass $1M_\odot$.

(a) Show that this cloud is gravitationally stable; that is, it could be in hydrostatic equilibrium.

Solution: The Jeans mass for this density ($\rho = \mu n = 3.3 \times 10^{-18}$ g/cm^3), composition ($\mu = 3.347 \times 10^{-24}$ g for H_2) and temperature is

$$M_J = \left(\frac{kT}{\mu G}\right)^{3/2} \left(\frac{3}{4\pi\rho}\right)^{1/2} = 4.2 \times 10^{33}\ \text{g} = 2.1 M_\odot > M_{\text{cloud}}$$

so it will not collapse under its own weight.

(b) To what temperature would it have to cool in order to become gravitationally unstable?

Solution: If the Jeans mass were less than or equal to the cloud mass then it would be unstable, so we find the critical temperature from the inequality

$$M_J = \left(\frac{kT}{\mu G}\right)^{3/2} \left(\frac{3}{4\pi\rho}\right)^{1/2} \leq 1M_\odot$$

$$T \leq \left(\frac{4\pi G^3 \mu^4 n M^2}{3k^3}\right)^{1/3} = 60.6\ \text{K}$$

(c) Suppose the cloud radiates like a blackbody and stays uniform in temperature and constant in size as it does so. How long, in hours, will it last before it becomes gravitationally unstable?

Hint: Recall that the luminosity of a spherical blackbody with radius R is $L = 4\pi R^2 \sigma T^4$ and that the thermal energy of N atoms at temperature T is $E = 3/2\ NkT$. Note that the factor in the thermal energy expression is actually 5/2 for a gas of diatomic molecules like hydrogen if the temperature is high enough (but 100 K is not high enough).

Solution: Since

$$E = \frac{3}{2}NkT = \frac{3}{2}\frac{M}{\mu}kT \qquad\qquad L = -\frac{dE}{dt}$$

we can derive, separate, and integrate the following differential equation:

$$\frac{dE}{dt} = \frac{3}{2}\frac{M}{\mu}k\frac{dT}{dt} = -L = -4\pi R^2 \sigma T^4$$

$$-\frac{3kM}{8\pi R^2 \sigma \mu T^4}\frac{dT}{dt} = 1$$

$$-\frac{3kM}{8\pi R^2 \sigma \mu}\int_{T_0}^{T_1} \frac{dT}{T^4} = \int_{t_0}^{t_1} dt$$

$$-\frac{3kM}{8\pi R^2 \sigma \mu}\left[-\frac{1}{3T^3}\right]_{T_0}^{T_1} = t_1 - t_0 = \Delta t$$

$$\Delta t = \frac{kM}{8\pi R^2 \sigma \mu}\left(\frac{1}{T_1^3} - \frac{1}{T_0^3}\right)$$

Since $T_0 = 100$ K and $T_1 = 61.7$ K and the cloud radius is

$$R = (3M/4\pi\rho)^{1/3} = 5.2 \times 10^{16}\ \text{cm}$$

we obtain

$$\Delta t = 7.54 \times 10^4 \text{ sec} \approx 20.9 \text{ hr}$$

This is a short time on the scale of the dimensions and internal velocities of such a cloud, which shows that radiative cooling is efficient enough that a molecular-cloud fragment can find itself suddenly unstable.

4. **Extinction toward open clusters**: On the course website, you will find photometric data files on the zero-age main sequence (ZAMS) derived from observations of nearby stars, and six additional open clusters.

   (a) Choose two of the clusters, and for each cluster, determine the color excesses $E(B-V)$ and $E(U-B)$ by plotting their $U-B$ colors as functions of their $B-V$ colors on the same plot as $U-B$ vs. $B-V$ for the ZAMS, and shifting the cluster data until they line up best with the ZAMS. Estimate the uncertainties in both quantities by the range of each color excess over which the fit to the ZAMS is good.

   Solution: See the left panel in the figures at the end of these solutions and the Python notebook posted on the course website. Note that none of the corrections come out as nicely as the Hyades, which you did in recitation; these clusters are much more distant and their members are much fainter, so the magnitude measurements are fewer and contain larger uncertainties. The range of $E(B-V)$ over which the fits are about right is roughly ± 0.03 to ± 0.05 magnitudes. Results are listed in the table below.

   (b) Measurements on large collections of open clusters typically give $E(U-B)/E(B-V) = 0.72$, but with substantial scatter about this value that is partly due to observational uncertainties and partly due to real differences between the extinction in various directions. How does this value compare with your results? Are they the same within uncertainties or does it look as if the extinction toward your cluster really has a different ratio of color excesses?

   Solution: Five of the clusters are indistinguishable from $E(U-B)/E(B-V) = 0.72$. Only NGC 7790 came out significantly different, with $E(U-B)/E(B-V) = 0.42$ giving the best fit. Thus, we would not be surprised for this cluster to demand a value of R different from 3.06.

5. **Measuring distances to open clusters by main-sequence fitting**:

   (a) For each of the two clusters you used in the previous problem, calculate extinction-corrected $B-V$ colors and V magnitudes for the stellar members, $(B-V)_0$, and $m_{V0} = V_0$. Plot (with Python or similar) V_0 as a function of $(B-V)_0$, the extinction-corrected H-R diagrams.

   Solution: Computed by

   $$(B-V)_0 = B - V - E(B-V)$$
   $$V_0 = V - R \cdot E(B-V)$$

   Details are available in `HW6.ipynb`. See also the table below and the center panels of the below figures.

   (b) The difference along the V axis between the main sequences of your clusters and the ZAMS are the same as the distance moduli of your clusters. Plot (again, using Python or something similar) each extinction-corrected HR diagram along with that of the ZAMS and shift the cluster's points until the cluster's main sequence and the ZAMS line up well. What is the distance modulus (magnitudes) and distance (in parsecs) of each cluster?

Solution: Computed from

$$DM = m - M = V_0 - M_V$$

where the values of M_V are those on the ZAMS curve. Details are in `HW6.ipynb`. See also the table below and the rightmost panels of the below figures. To compute the distance in parsecs:

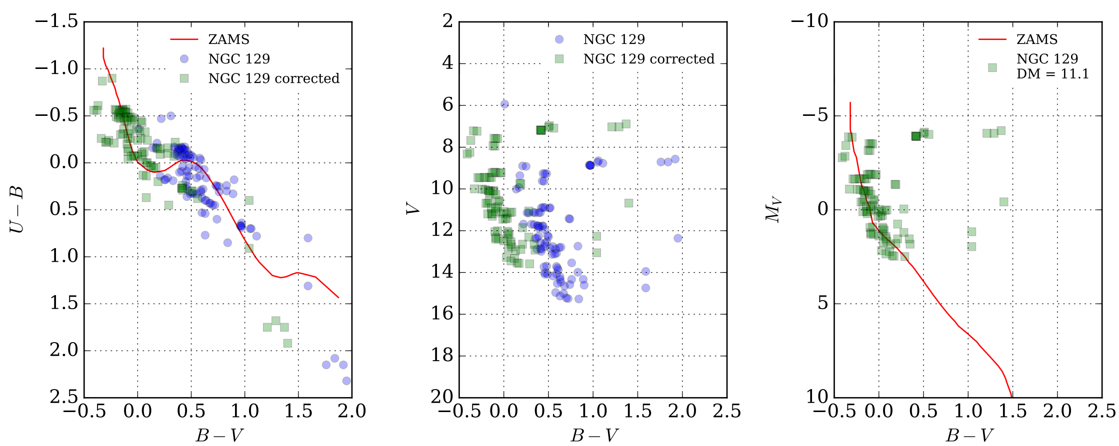
$$DM = 5 \log (r/10 \text{ pc})$$

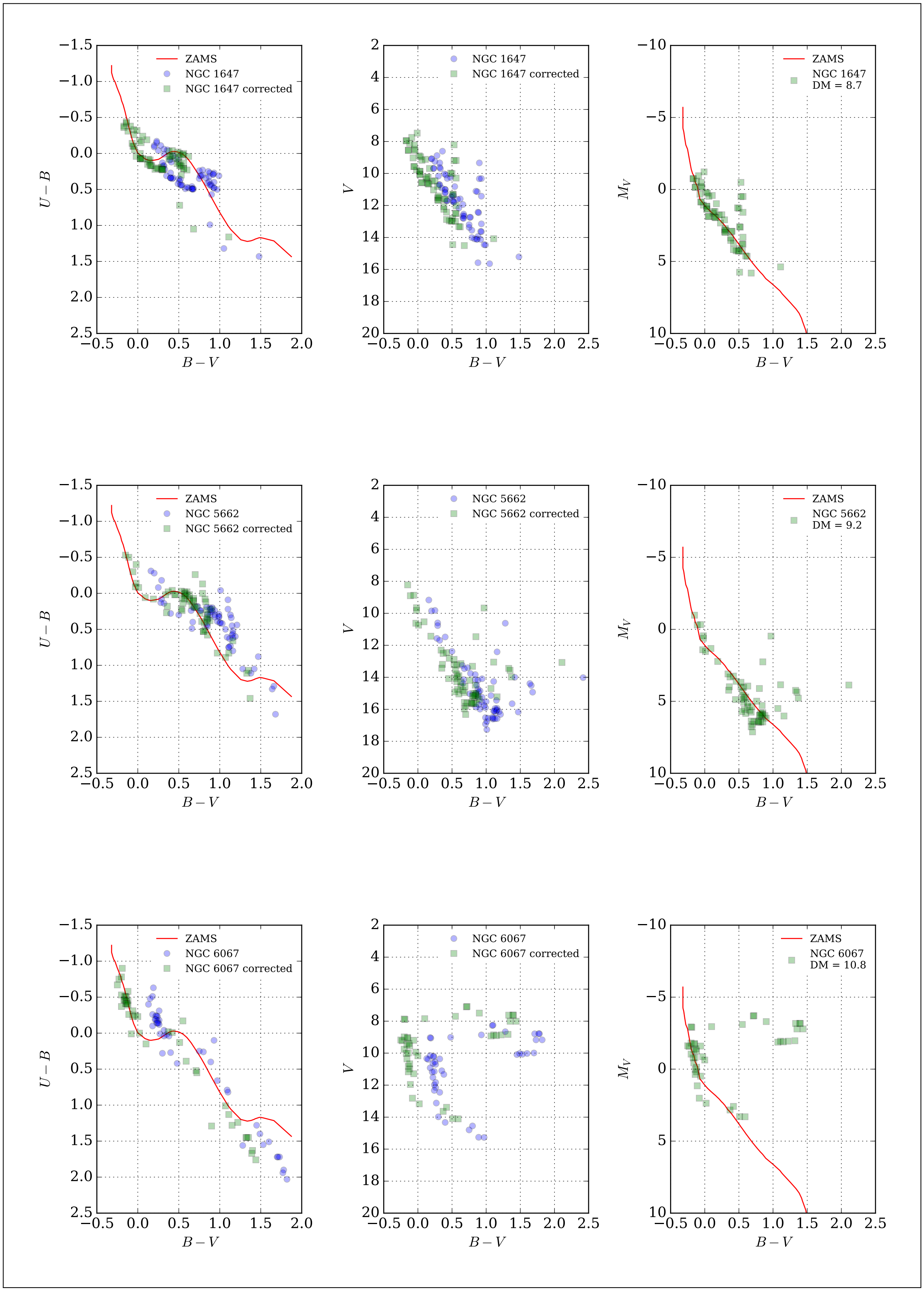
$$r = 10^{DM/5+1} \text{ pc}$$

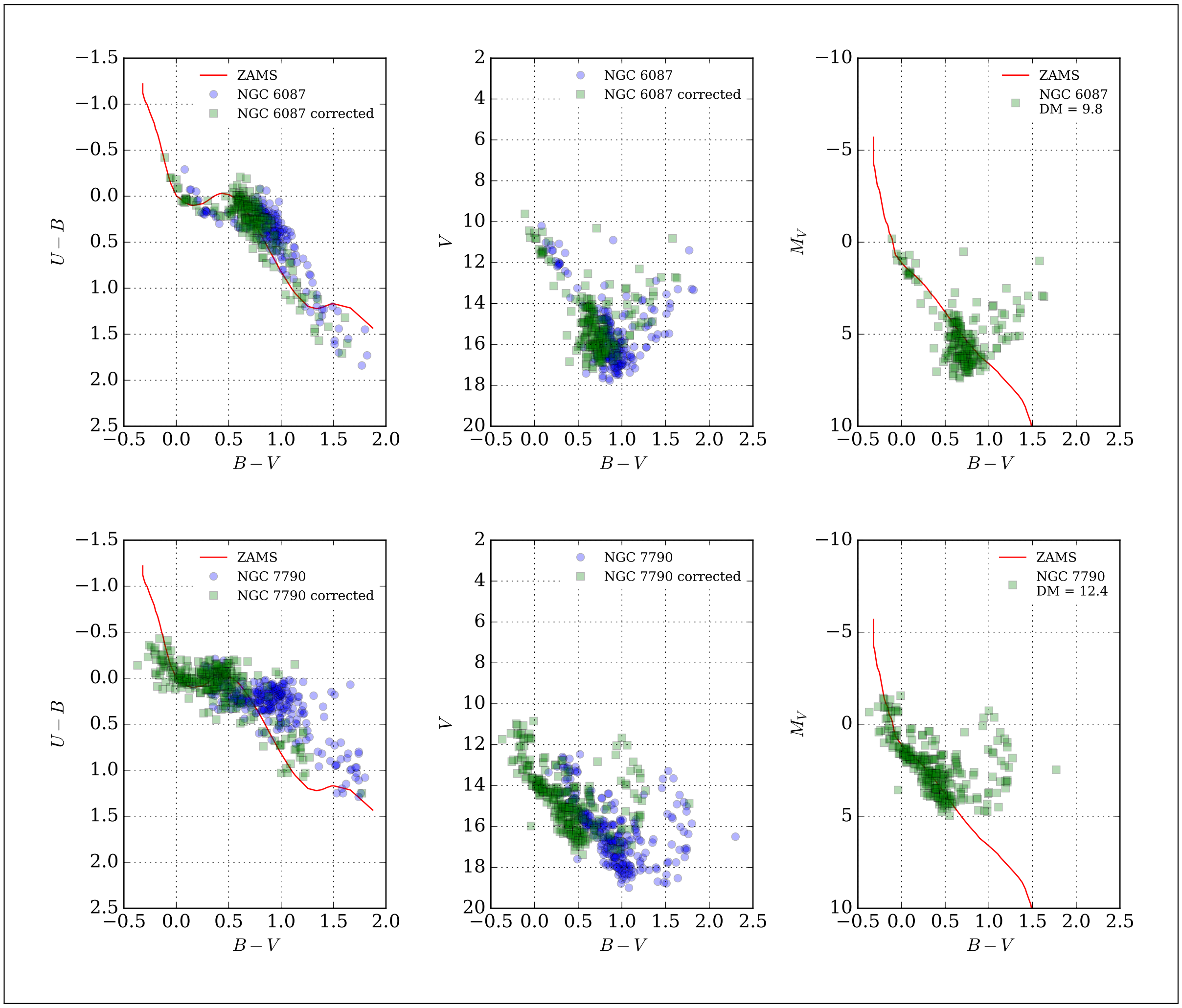
Solution:

Tabulated values and plots for the previous two problems:

Cluster	$E(B-V)$	$E(U-B)$	A_v	$V_0 - M_v$	r [pc]
NGC 129	0.55	0.4	1.7	11.1	1660
NGC 1647	0.37	0.27	1.1	8.7	550
NGC 5662	0.31	0.22	0.9	9.2	692
NGC 6067	0.38	0.27	1.2	10.8	1445
NGC 6087	0.19	0.13	0.6	9.8	912
NGC 7790	0.53	0.22	1.6	12.4	3020



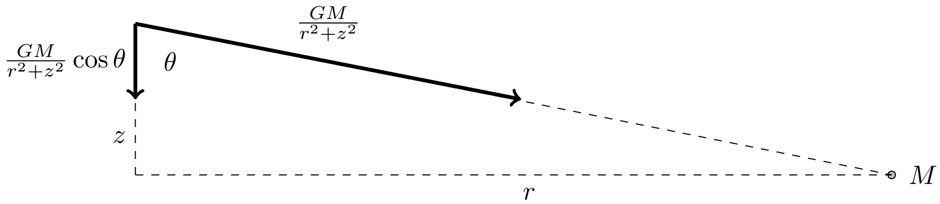

ZAMS
NGC 1647
NGC 1647 corrected
NGC 1647
DM = 8.7
ZAMS
NGC 5662
NGC 5662 corrected
NGC 5662
DM = 9.2
ZAMS
NGC 6067
NGC 6067 corrected
NGC 6067
DM = 10.8
U − B
V
M_V
B − V



6. Disks generally decrease strongly in density from the inside out. Eventually they cease to be self-gravitating, changing their structure. Consider the outer reaches of a galactic disk at a distance at which the gravitational forces are dominated by the central part of the galaxy (mass M), which you can assume to be spherical. The disk is still supported centrifugally in the radial direction and hydrostatically in the vertical direction, but now the weight of a test particle is determined by the vertical component of the force from the galactic center (like the situation of a protoplanetary disk around a young star).

   (a) Under the assumptions $r \gg z$ and that the vertical component of random stellar velocities v_z is independent of z, solve the equation of hydrostatic equilibrium at radius r for the mass density ρ as a function of z. What is the density scale height?

   Solution:
   The geometry is shown below:



$$\frac{dP}{dz} = -\rho g_z$$
$$v_z^2 \frac{d\rho}{dz} = -\rho \frac{G}{z^2+r^2}\cos\theta = -\rho\frac{G}{z^2+r^2}\frac{z}{\sqrt{z^2+r^2}} \approx -\rho\frac{GM}{r^3}z \quad \text{for } z \ll r$$

This can be separated and integrated from the mid-plane at $z = 0$, where the density is ρ_0, up to vertical height z:

$$\frac{dP}{dz} = -\rho\frac{GM}{r^3}z$$
$$\int_{\rho_0}^{\rho(z)} \frac{d\rho'}{\rho'} = -\frac{GM}{v_z^2 r^3}\int_0^z z'\, dz'$$
$$\ln\rho(z) - \ln\rho_0 = \frac{GM}{2v_z^2 r^3}z^2 = \frac{z^2}{H^2} \qquad H = \sqrt{\frac{2v_z^2 r^3}{GM}}$$
$$\rho(z) = \rho_0 e^{-(z/H)^2}$$

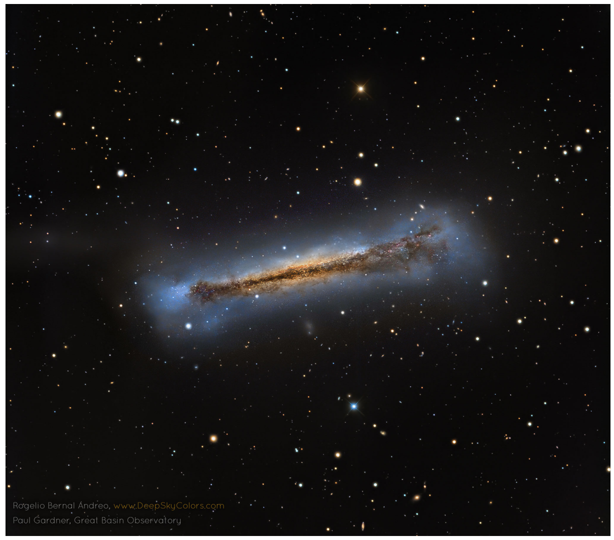
Vertically the disk follows a Gaussian functional form.
The scale height is $H = \sqrt{2v_z^2 r^3/GM}$.

(b) Under the assumption that the vertical component of random stellar velocities v_z is also independent of r, show that the disk is flared, i.e., the scale height increases with increasing radius.

Solution:
If v_z does not vary with r then $H \propto r^{3/2}$. So the disk is flared because H increases sharply with r.

(c) Look online for an image of the galaxy NGC 3628. Might this be a good example of a flared galactic disk? Why or why not? Discuss briefly.

Solution:
This picture of NGC 3628, the "Hamburger Galaxy," comes from APOD (Sept. 5, 2018).



The disk is flared, but from the heavy dust obscuration and bright starlight it looks as if there is still a lot of mass out at the edge. So the conditions might not be what we have been considering when solving the problem. The disk could be flared for other reasons, such as the perturbations produced by the nearby galaxies M65 and M66.