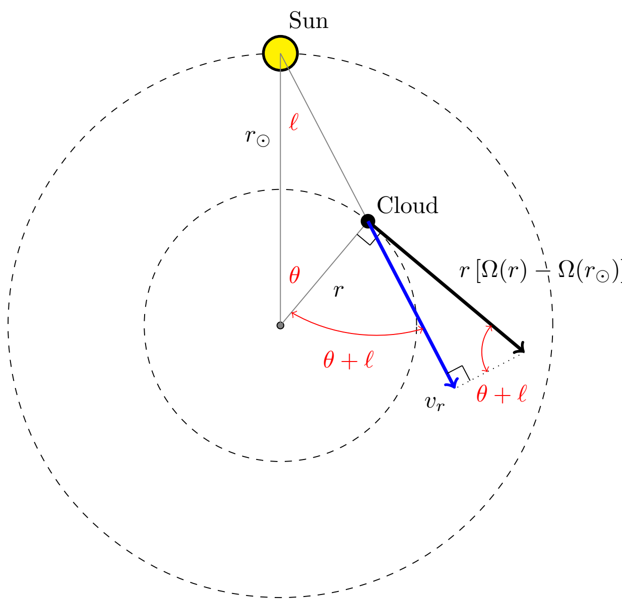
1. **The Milky Way's rotation curve**
   (a) Argue that in the frame of reference revolving around the Galactic Center with angular speed $\Omega(r_\odot)$, the circular velocity of a gas cloud in the plane of the Milky Way at galactocentric distance r is $r\left[\Omega(r) - \Omega(r_\odot)\right]$.
   (b) Use the diagram below to justify the various angular identifications to show that the radial velocity of the gas cloud is given by
   $$v_r = r\left[\Omega(r) - \Omega(r_\odot)\right]\sin(\theta + \ell)$$
   where ℓ is the Galactic longitude of the cloud.



   (c) Use the law of sines to show that
   $$v_r = r_\odot\left[\Omega(r) - \Omega(r_\odot)\right]\sin\ell \qquad (1)$$
   (d) For a given Galactic longitude ℓ, the only thing that can vary in Equation 1 is the galactocentric distance of the gas cloud. Argue that if $\Omega(r)$ is a monotonically decreasing function of r, then v_r acquires its maximum positive value (for $0^\circ \leq \ell \leq 90^\circ$) when r corresponds to the radius of the circular orbit tangent to the line of sight.
   (e) Thus, show that the orbital speed at the tangent point (the speed we would want to use with Newton's laws or Kepler's laws, to work out masses) is given by
   $$v(r) = r\Omega(r) = v_{r,\max} + r_\odot\Omega(r_\odot)\sin\ell$$
   where everything on the right hand side can be obtained from observations.

2. **Galactocentric distance and the distance ambiguity**: If we know $r_\odot$, $\Omega(r_\odot)$, and the functional form of $\Omega(r)$, and we measure v_r, when we point a radio telescope in direction ℓ, the equation you derived in Problem 1 allows us to deduce a cloud's galactocentric distance r.
   (a) Derive from this an expression for r. You can assume that the rotational tangential velocities in the disk are nearly constant with r.
   (b) With a glance at the figure in Problem 1, show that if $r < r_\odot$ the line of sight generally intersects the circle of radius r at two points: a near point and a far point. This is the distance ambiguity. Show that the distance ambiguity does not arise if $r > r_\odot$.

3. The **virial theorem** for a spherical self-gravitating star cluster in "thermal equilibrium" (treating the stars like particles in a gas) states that the total kinetic energy of N stars with typical random speed V is equal to minus one half of the total gravitational potential energy. If R is the average separation between any two stars (assumed to be of equal mass m), then the gravitational potential energy of the pair is $-Gm^2/R$. Note that there are $N(N-1)/2$ possible pairings of the stars.
   (a) Taking R to also be the size of the core of the cluster (its "core radius"), show that the typical escape speed from the cluster is
   $$v_{\text{esc}} = \sqrt{\frac{2G(N-1)m}{R}}$$
   (b) Show that $v_{\text{esc}} = 2V$.
   (c) Estimate R numerically for star clusters with $N = 1000$, $m = 1M_\odot$, and $V = 1$ km/s (open cluster) and with $N = 10^6$, $m = 0.5M_\odot$, and $V = 20$ km/s (globular cluster).

4. **Velocity dispersion**
   (a) Consider a spherical stellar cluster with $N \gg 1$ members and core radius R which on average is at rest with respect to us. The members of the cluster can be considered to have typical mass m and typical average value $\overline{v^2}$ for the square of the velocity relative to the average. The square root of this quantity is called the **velocity dispersion**. Use the virial theorem to find an expression for the total mass of the cluster, in terms of the core radius and the velocity dispersion.
   (b) A star with speed v and velocity vector pointing at an angle θ with respect to the line of sight would be observed with radial velocity $v_r = v\cos\theta$. Suppose that the directions of motion of stars are random and that the three-dimensional average value of the square of $\cos\theta$ (averaged over solid angle) is
   $$\overline{\cos^2\theta} = \frac{\int_0^{2\pi} d\phi \int_0^{\pi} \cos^2\theta \ \sin\theta d\theta}{\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta}$$
   Show that the mass of the cluster can be determined from observations of radial velocity and core radius as
   $$M = \frac{6R\overline{v_r^2}}{G}$$

5. The azimuthally averaged surface brightness of the disk of a spiral galaxy is given in terms of distance R within the disk from the center of the galaxy by
   $$\mathcal{L}(R) = \mathcal{L}(0)e^{-R/R_0}$$
   Consider a nearly face-on spiral galaxy for which the "scale length" of the disk surface brightness is $R_0 = 2$ kpc, and for which the mass-to-light ratio is 2 at this galactocentric radius. Suppose you have shown that the rotation curve is flat over the galactocentric radii $2 - 12$ kpc, so that you might suggest a spherical halo with density
   $$\rho(r) = \rho_0 \left(\frac{r_0}{r}\right)^2$$
   to dominate the mass in this range, where r is the usual spherical coordinate. Assume for simplicity that the core radius of the halo r_0 is equal to the scale length R_0 of the disk surface brightness.
   (a) Show that the effective mass per unit area of the disk due to the spherical halo is
   $$\mu(R) = \frac{\pi\rho_0 r_0^2}{R}$$
   by integrating the density along the direction z perpendicular to the disk. *Hint*: Note that $r = \sqrt{R^2 + z^2}$ and use a trigonometric substitution to simplify the resulting integral.

(b) Plot (with Python or similar) the mass-to-light ratio $\mu(R)/\mathcal{L}(R)$ for $R = 2 - 12$ kpc. What is the value of the mass-to-light at the largest radii? How does this compare to the mass-to-light ratio in the solar neighborhood?

(c) By adjustment of the halo core radius r_0, is it possible to make the mass-to-light ratio constant? Why or why not?

