1. **The Milky Way's rotation curve**

   (a) Argue that in the frame of reference revolving around the Galactic Center with angular speed $\Omega(r_\odot)$, the circular velocity of a gas cloud in the plane of the Milky Way at galactocentric distance r is $r\,[\Omega(r) - \Omega(r_\odot)]$.

   Solution: Refer to the above figure and consider a gas cloud at galactocentric distance r that rotates with angular speed $\Omega(r)$. This cloud will seem to an observer at galactocentric distance $r_\odot$ to be rotating at angular speed

   $$\Omega(r) - \Omega(r_\odot)$$

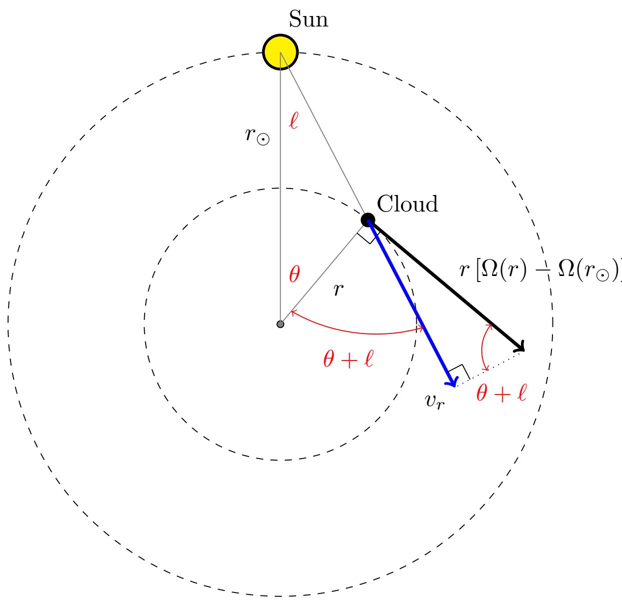
   The speed of the cloud in this reference frame is

   $$v = r\,[\Omega(r) - \Omega(r_\odot)]$$

   (b) Use the diagram below to justify the various angular identifications to show that the radial velocity of the gas cloud is given by

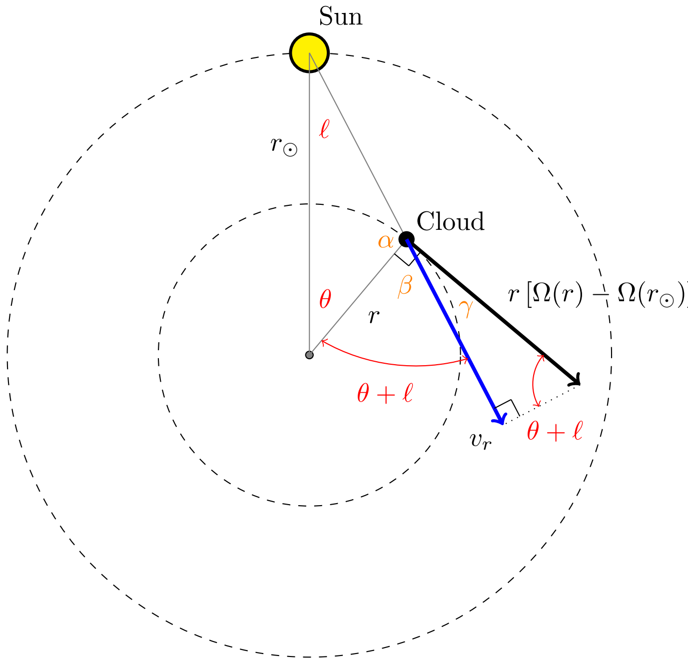
   $$v_r = r\,[\Omega(r) - \Omega(r_\odot)]\sin(\theta + \ell)$$

   where ℓ is the Galactic longitude of the cloud.



   Solution: The direction of its velocity is tangent to the circle of radius r and thus perpendicular to the radius drawn from the Galactic center. We need to know the component of this velocity along the line of sight since this is the radial velocity v_r measured by the Doppler shift.

   We will begin by calculating the angles α, β, and γ in the diagram:



First α: since the sum of the interior angles of a triangle is π,

$$\alpha = \pi - \theta - \ell$$

Next, β: clearly α and β are supplementary angles so

$$\beta = \pi - \alpha = \theta + \ell$$

Last is γ: since the velocity vector and radial vector are perpendicular, $\beta + \gamma = \pi/2$, so

$$\gamma = \frac{\pi}{2} - \beta = \frac{\pi}{2} - \theta - \ell$$

The line-of-sight component of the velocity (the "radial velocity" in with respect to the observer) is

$$\begin{aligned} v_r &= v\cos\gamma = v\cos\left(\frac{\pi}{2} - \theta - \ell\right) = v\sin\left(\theta + \ell\right) \\ &= r\left[\Omega(r) - \Omega(r_\odot)\right]\sin\left(\theta + \ell\right) \end{aligned}$$

(c) Use the law of sines to show that

$$v_r = r_\odot\left[\Omega(r) - \Omega(r_\odot)\right]\sin\ell \tag{1}$$

Solution: Next, apply the law of sines to the triangle defined by the Sun, the Galactic Center, and the cloud:

$$\frac{\sin\alpha}{r_\odot} = \frac{\sin\ell}{r}$$

However,

$$\sin\alpha = \sin\left(\pi - \theta - \ell\right) = \sin\left(\theta + \ell\right)$$

Substituting back into the law of sines expression and cross- multiplying gives

$$r \sin(\theta + \ell) = r_\odot \sin \ell$$

Plugging this back into the expression for v_r gives

$$v_r = r_\odot [\Omega(r) - \Omega(r_\odot)] \sin \ell$$

(d) For a given Galactic longitude ℓ, the only thing that can vary in Equation 1 is the galactocentric distance of the gas cloud. Argue that if $\Omega(r)$ is a monotonically decreasing function of r, then v_r acquires its maximum positive value (for $0° \leq \ell \leq 90°$) when r corresponds to the radius of the circular orbit tangent to the line of sight.

Solution: Now, by assumption, $\Omega(r)$ decreases monotonically with r (required for a flat rotation curve). Thus, the largest value of v_r in a given line of sight comes from the smallest r included in the line of sight, which in turn is the radius of the smallest circle intersected by the line of sight.

(e) Thus, show that the orbital speed at the tangent point (the speed we would want to use with Newton's laws or Kepler's laws, to work out masses) is given by

$$v(r) = r\Omega(r) = v_{r,\max} + r_\odot \Omega(r_\odot) \sin \ell$$

where everything on the right hand side can be obtained from observations.

Solution: The line of sight is tangent to the smallest circle it intersects, so

$$\theta + \ell = \frac{\pi}{2}$$

for the cloud with the maximum radial velocity in this direction. According to the expression we derived above, this is

$$v_{r,\max} = r_{\min} [\Omega(r_{\min}) - \Omega(r_\odot)] \sin\left(\frac{\pi}{2}\right) = r_{\min} [\Omega(r_{\min}) - \Omega(r_\odot)]$$

where $r_{\min} = r_\odot \sin \ell$. For any line of sight we can write the circular velocity of clouds at the tangent point as

$$v(r_{\min}) = r_{\min}\Omega(r_{\min}) = v_{r,\max} + r_{\min}\Omega(r_\odot) = v_{r,\max} + r_\odot \Omega(r_\odot) \sin \ell$$

The maximum radial velocity comes from the maximum Doppler shift in the spectrum. The Galactic longitude ℓ is the angle bretween the light of sight and the cloud. The Galactocentric distance $r_\odot$ comes from proper motion measurements of objects near the Galactic center. And the angular speed in orbit comes from observations of stellar velocities in the Solar neighborhood and determination of the Oort constants A and B. Therefore, everything in the above expression is observable.

2. **Galactocentric distance and the distance ambiguity**: If we know $r_\odot$, $\Omega(r_\odot)$, and the functional form of $\Omega(r)$, and we measure v_r, when we point a radio telescope in direction ℓ, the equation you derived in Problem 1 allows us to deduce a cloud's galactocentric distance r.

(a) Derive from this an expression for r. You can assume that the rotational tangential velocities in the disk are nearly constant with r.

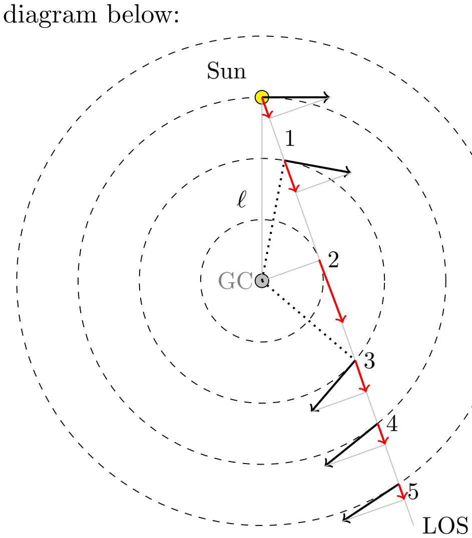
Solution: Since $\Omega = v/r$, then

$$\begin{aligned} v_r &= r_\odot[\Omega(r) - \Omega(r_\odot)]\sin\ell \\ &= r_\odot\left[\frac{v}{r} - \frac{v}{r_\odot}\right]\sin\ell \\ r &= r_\odot\frac{v\sin\ell}{v_r + v\sin\ell} \end{aligned}$$

Everything on the right-hand side of this expression is measurable. An observation of the radial Doppler velocity v_r and the longitude ℓ allows a determination of a cloud's galactocentric radius r.

(b) With a glance at the figure in Problem 1, show that if $r < r_\odot$ the line of sight generally intersects the circle of radius r at two points: a near point and a far point. This is the distance ambiguity. Show that the distance ambiguity does not arise if $r > r_\odot$.

Solution: Consider the diagram below:



The line of sight from the Sun intersects each inner orbit twice, for example at the points labeled 1 and 3.

Notice that the angle formed by the line of sight and the line from GC to 1 (defined by the triangle GC-1-2) is congruent to the angle between the line of sight and the line from GC to 3 (defined by the triangle GC-3-2). This angle, labeled $\theta + \ell$ in the figure from Problem 1, determines the component of the velocity v along the line of sight,

$$v_r = v\sin(\theta + \ell)$$

and so both points will have the same Doppler velocity v_r. Therefore we have no easy way to determine if the point is at location 1 or 3 unless we can observe the proper motion, which may not be practical — it takes millions of years for the objects at this radius to move appreciably with respect to our line of sight.

Thus, we have to use more indirect measurements to tell whether or not we're looking at point 1 or 3. For example, if the cloud has a visible-light counterpart it's probably at point 1, because galactic dust would easily obscure a more distant object at point 3. Alternatively, if the angular size is large it's also more likely to be at point 1 than 3.

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3. The **virial theorem** for a spherical self-gravitating star cluster in "thermal equilibrium" (treating the stars like particles in a gas) states that the total kinetic energy of N stars with typical random speed V is equal to minus one half of the total gravitational potential energy. If R is the average separation between any two stars (assumed to be of equal mass m), then the gravitational potential energy of the pair is $-Gm^2/R$. Note that there are $N(N-1)/2$ possible pairings of the stars.

   (a) Taking R to also be the size of the core of the cluster (its "core radius"), show that the typical escape speed from the cluster is

$$v_{\text{esc}} = \sqrt{\frac{2G(N-1)m}{R}}$$

Solution: Each star is a member of $N-1$ pairs, so the binding energy of one star is $N-1$ times the gravitational potential energy of one pair:

$$U_1 = -\frac{G(N-1)m^2}{R}$$

For the star to escape its total energy must be > 0, so the kinetic energy $mv_{\text{esc}}^2/2$ must be $\geq -U_1$. Thus

$$\frac{1}{2}mv_{\text{esc}}^2 \geq \frac{G(N-1)m^2}{R}$$
$$v_{\text{esc}} \geq \sqrt{\frac{2G(N-1)m}{R}}$$

   (b) Show that $v_{\text{esc}} = 2V$.

Solution: The total gravitational potential energy of the cluster is

$$U = -\frac{GN(N-1)m^2}{2R}$$

From the virial theorem,

$$K = -\frac{U}{2}$$
$$\frac{NmV^2}{2} = \frac{GN(N-1)m^2}{4R}$$
$$V^2 = \frac{G(N-1)m}{2R}$$

Therefore,

$$v_{\text{esc}}^2 = \frac{2G(N-1)m}{R}$$
$$v_{\text{esc}} = 2V$$

   (c) Estimate R numerically for star clusters with $N = 1000$, $m = 1M_\odot$, and $V = 1$ km/s (open cluster) and with $N = 10^6$, $m = 0.5M_\odot$, and $V = 20$ km/s (globular cluster).

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Solution:

$$
\begin{aligned}
R &= \frac{G(N-1)m}{2V^2} \\
&= 6.7 \times 10^{18}\ \text{cm} = 2.2\ \text{pc} && \text{(open cluster)} \\
&= 8.3 \times 10^{18}\ \text{cm} = 2.7\ \text{pc} && \text{(globular cluster)}
\end{aligned}
$$

4. **Velocity dispersion**

(a) Consider a spherical stellar cluster with $N \gg 1$ members and core radius R which on average is at rest with respect to us. The members of the cluster can be considered to have typical mass m and typical average value $\overline{v^2}$ for the square of the velocity relative to the average. The square root of this quantity is called the **velocity dispersion**. Use the virial theorem to find an expression for the total mass of the cluster, in terms of the core radius and the velocity dispersion.

Solution: For a cluster of N objects of mass m moving at speed $\sqrt{\overline{v^2}}$ and separated by R on average, the total kinetic energy and gravitational potential energy is

$$K = \frac{1}{2}Nm\overline{v^2} \qquad\qquad U = -\frac{GN(N-1)m^2}{2R}$$

The latter formula is obtained from the fact that N objects can be arranged in $N(N-1)/2$ pairs. The virial theorem $K = -U/2$ implies that

$$
\begin{aligned}
\frac{1}{2}Nm\overline{v^2} &= \frac{GN(N-1)m^2}{4R} \\
\overline{v^2} &= \frac{G(N-1)m}{2R}
\end{aligned}
$$

If the number of objects is very large then $N - 1 \approx N$ and the total mass of the cluster is $M = Nm$, so

$$M = \frac{2R\overline{v^2}}{G}$$

(b) A star with speed v and velocity vector pointing at an angle θ with respect to the line of sight would be observed with radial velocity $v_r = v\cos\theta$. Suppose that the directions of motion of stars are random and that the three-dimensional average value of the square of $\cos\theta$ (averaged over solid angle) is

$$\overline{\cos^2\theta} = \frac{\int_0^{2\pi} d\phi \int_0^{\pi} \cos^2\theta\ \sin\theta d\theta}{\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta}$$

Show that the mass of the cluster can be determined from observations of radial velocity and core radius as

$$M = \frac{6R\overline{v_r^2}}{G}$$

Solution: The cluster mass can be expressed in terms of observable parameters as

$$M = \frac{2R\overline{v_r^2}}{G\overline{\cos^2\theta}}$$

where

$$\overline{\cos^2\theta} = \frac{\int_0^{2\pi} d\phi \int_0^{\pi} \cos^2\theta \ \sin\theta d\theta}{\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta} = \frac{2\pi \int_{-1}^{1} -u^2 du}{4\pi} = \frac{1}{3}$$

By the same token,

$$\overline{\sin^2\theta} = \frac{2}{3}$$

5. The azimuthally averaged surface brightness of the disk of a spiral galaxy is given in terms of distance R within the disk from the center of the galaxy by

$$\mathcal{L}(R) = \mathcal{L}(0)e^{-R/R_0}$$

Consider a nearly face-on spiral galaxy for which the "scale length" of the disk surface brightness is $R_0 = 2$ kpc, and for which the mass-to-light ratio is 2 at this galactocentric radius. Suppose you have shown that the rotation curve is flat over the galactocentric radii $2 - 12$ kpc, so that you might suggest a spherical halo with density

$$\rho(r) = \rho_0 \left(\frac{r_0}{r}\right)^2$$

to dominate the mass in this range, where r is the usual spherical coordinate. Assume for simplicity that the core radius of the halo r_0 is equal to the scale length R_0 of the disk surface brightness.

(a) Show that the effective mass per unit area of the disk due to the spherical halo is

$$\mu(R) = \frac{\pi\rho_0 r_0^2}{R}$$

by integrating the density along the direction z perpendicular to the disk. *Hint*: Note that $r = \sqrt{R^2 + z^2}$ and use a trigonometric substitution to simplify the resulting integral.

Solution: First integrate the density along z:

$$\mu(R) = \int_{-\infty}^{\infty} \rho(R, z) \ dz = \rho_0 r_0^2 \int_{-\infty}^{\infty} \frac{dz}{R^2 + z^2} = \frac{\rho_0 r_0^2}{R} \int_{-\infty}^{\infty} \frac{dz/R}{1 + (z/R)^2}$$

Substitute $\tan\theta = z/R$, such that

$$\frac{z}{R} = \tan\theta \qquad \frac{dz}{R} = \frac{d}{d\theta}\tan\theta = (1 + \tan^2\theta)d\theta$$

The range in this variable is

$$z \in (-\infty, \infty) \qquad \theta = \arctan(z/R) \in (-\pi/2, \pi/2)$$

Thus,

$$\mu(R) = \frac{\rho_0 r_0^2}{R} \int_{-\pi/2}^{\pi/2} \frac{(1 + \tan^2\theta)d\theta}{1 + \tan^2\theta} = \frac{\rho_0 r_0^2}{R} \int_{-\pi/2}^{\pi/2} d\theta = \frac{\pi\rho_0 r_0^2}{R}$$

(b) Plot (with Python or similar) the mass-to-light ratio $\mu(R)/\mathcal{L}(R)$ for $R = 2 - 12$ kpc. What is the value of the mass-to-light at the largest radii? How does this compare to the mass-to-light ratio in the solar neighborhood?

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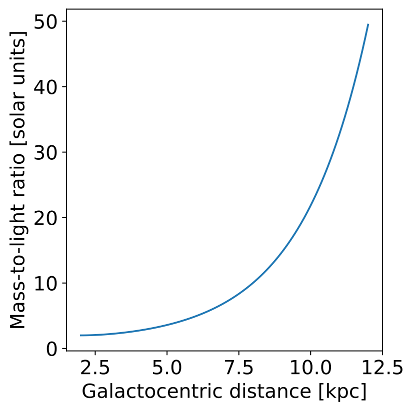
Solution: First we need to work out the ratio of the constant factors $\mathcal{L}(0)$ and ρ_0. We have

$$\frac{M}{L}(R) = \frac{\mu(R)}{\mathcal{L}(R)} = \frac{\pi\rho_0 r_0^2/R}{\mathcal{L}(0)e^{-R/R_0}} = 2\frac{M_\odot}{L_\odot} \quad \text{at } R = 2 \text{ kpc, or}$$

$$\frac{\rho_0}{\mathcal{L}(0)} = 2\left(\frac{2 \text{ kpc}}{\pi \cdot (2 \text{ kpc})^2}\right) e^{-1}\frac{M_\odot}{L_\odot} = \frac{1}{\pi e}\frac{M_\odot}{L_\odot} \text{ kpc}^{-1}$$

$$\frac{M}{L}(R) = \frac{r_0^2/R}{e^{-R/R_0+1}}\frac{M_\odot}{L_\odot} \text{ kpc}^{-1}$$

This is plotted below. Note that in the outer reaches of the galaxy the mass-to-light ratio is about 10× that in the Solar neighborhood. This must mean that the stellar mass function there is biased toward stars much less luminous than those near us, or that other forms of dark matter are present.



(c) By adjustment of the halo core radius r_0, is it possible to make the mass-to-light ratio constant? Why or why not?

Solution: No. This would only work precisely if the mass and light distribution had the same functional form.