- 1. Disks generally decrease strongly in density from the inside out. Eventually they cease to be selfgravitating, changing their structure. Consider the outer reaches of a galactic disk at a distance at which the gravitational forces are dominated by the central part of the galaxy (mass M), which you can assume to be spherical. The disk is still supported centrifugally in the radial direction and hydrostatically in the vertical direction, but now the weight of a test particle is determined by the vertical component of the force from the galactic center (like the situation of a protoplanetary disk around a young star).
 - (a) Under the assumptions $r \gg z$ and that the vertical component of random stellar velocities v_z is independent of z, solve the equation of hydrostatic equilibrium at radius r for the mass density ρ as a function of z. What is the density scale height?



This can be separated and integrated from the mid-plane at z = 0, where the density is ρ_0 , up to vertical height z:

$$\frac{dP}{dz} = -\rho \frac{GM}{r^3} z$$

$$\int_{\rho_0}^{\rho(z)} \frac{d\rho'}{\rho'} = -\frac{GM}{v_z^2 r^3} \int_0^z z' \, dz'$$

$$\ln \rho(z) - \ln \rho_0 = \frac{GM}{2v_z^2 r^3} z^2 = \frac{z^2}{H^2} \qquad H = \sqrt{\frac{2v_z^2 r^3}{GM}}$$

$$\rho(z) = \rho_0 e^{-(z/H)^2}$$

Vertically the disk follows a Gaussian functional form. The scale height is $H = \sqrt{2v_z^2 r^3/GM}$.

(b) Under the assumption that the vertical component of random stellar velocities v_z is also independent of r, show that the disk is flared, i.e., the scale height increases with increasing radius.

Solution:

If v_z does not vary with r then $H \propto r^{3/2}$. So the disk is flared because H increases sharply with r.

(c) Look online for an image of the galaxy NGC 3628. Is this a good example of a flared galactic disk? Why or why not? Discuss briefly.



2. The **virial theorem** for a spherical self-gravitating stellar cluster in "thermal equilibrium" (treating the stars like particles in a gas) states that the total kinetic energy of N stars with typical random speed V is equal to minus one half of the total gravitational potential energy. If R is the average separation between any two stars (assumed to be of equal mass m), then the gravitational potential energy of the pair is $-Gm^2/R$. Note that there are N(N-1)/2 possible pairings of the stars.

(a) Taking R to also be the size of the core of the cluster (its "core radius"), show that the typical escape speed from the cluster is

$$v_{\rm esc} = \sqrt{\frac{2G(N-1)m}{R}}$$

Solution: Each star is a member of N - 1 pairs, so the binding energy of one star is N - 1 times the gravitational potential energy of one pair:

$$U_1 = -\frac{G(N-1)m^2}{R}$$

For the star to escape its total energy must be > 0, so the kinetic energy $mv_{\rm esc}^2/2$ must be $\geq -U_1$. Thus

$$\frac{1}{2}mv_{\rm esc}^2 \ge \frac{G(N-1)m^2}{R}$$
$$v_{\rm esc} \ge \sqrt{\frac{2G(N-1)m}{R}}$$

(b) Show that $v_{\text{esc}} = 2V$.

Solution: The total gravitational potential energy of the cluster is	
$U = -\frac{GN(N-1)m^2}{2R}$	
From the virial theorem,	
$K = -\frac{U}{2}$ $\frac{NmV^2}{2} = \frac{GN(N-1)m^2}{4R}$ $V^2 = \frac{G(N-1)m}{2R}$	

Therefore,

$$v_{\rm esc}^2 = \frac{2G(N-1)m}{R}$$
$$v_{\rm esc} = 2V$$

(c) Estimate R numerically for star clusters with N = 1000, $m = 1M_{\odot}$, and V = 1 km/s (open cluster) and with $N = 10^6$, $m = 0.5M_{\odot}$, and V = 20 km/s (globular cluster).

Solution:			
	$R = \frac{G(N-1)m}{2V^2}$		
	$= 6.7 \times 10^{18} \text{ cm} = 2.2 \text{ pc}$	(open cluster)	
	$= 8.3 \times 10^{18} \text{ cm} = 2.7 \text{ pc}$	(globular cluster)	

3. Velocity dispersion

(a) Consider a spherical stellar cluster with $N \gg 1$ members and core radius R which on average is at rest with respect to us. The members of the cluster can be considered to have typical mass m and typical average value $\overline{v^2}$ for the square of the velocity relative to the average. The square root of this quantity is called the **velocity dispersion**. Use the virial theorem to find an expression for the total mass of the cluster, in terms of the core radius and the velocity dispersion.

Solution: For a cluster of N objects of mass m moving at speed $\sqrt{v^2}$ and separated by R on average, the total kinetic energy and gravitational potential energy is

$$K = \frac{1}{2}Nm\overline{v^2} \qquad \qquad U = -\frac{GN(N-1)m^2}{2R}$$

The latter formula is obtained from the fact that N objects can be arranged in N(N-1)/2

pairs. The virial theorem K = -U/2 implies that

$$\frac{1}{2}Nm\overline{v^2} = \frac{GN(N-1)m^2}{4R}$$
$$\overline{v^2} = \frac{G(N-1)m}{2R}$$

If the number of objects is very large then $N-1 \approx N$ and the total mass of the cluster is M = Nm, so

$$M = \frac{2Rv^2}{G}$$

(b) A star with speed v and velocity vector pointing at an angle θ with respect to the line of sight would be observed with radial velocity $v_r = v \cos \theta$. Suppose that the directions of motion of stars are random and that the three-dimensional average value of the square of $\cos \theta$ (averaged over solid angle) is

$$\overline{\cos^2 \theta} = \frac{\int_0^{2\pi} d\phi \int_0^{\pi} \cos^2 \theta \, \sin \theta d\theta}{\int_0^{2\pi} d\phi \int_0^{\pi} \sin \theta d\theta}$$

Show that the mass of the cluster can be determined from observations of radial velocity and core radius as

$$M = \frac{6R\overline{v_i^2}}{G}$$

Solution: The cluster mass can be expressed in terms of observable parameters as

$$M = \frac{2R\overline{v_r^2}}{G\overline{\cos^2\theta}}$$

where

$$\overline{\cos^2\theta} = \frac{\int_0^{2\pi} d\phi \int_0^{\pi} \cos^2\theta \,\sin\theta d\theta}{\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta} = \frac{2\pi \int_{-1}^1 -u^2 du}{4\pi} = \frac{1}{3}$$

By the same token,

$$\overline{\sin^2 \theta} = \frac{2}{3}$$