1. The Milky Way's rotation curve

(a) Argue that in the frame of reference revolving around the Galactic Center with angular speed $\Omega(r_{\odot})$, the circular velocity of a gas cloud in the plane of the Milky Way at galactocentric distance r is $r [\Omega(r) - \Omega(r_{\odot})]$.

Solution: Refer to the above figure and consider a gas cloud at galactocentric distance r that rotates with angular speed $\Omega(r)$. This cloud will seem to an observer at galactocentric distance r_{\odot} to be rotating at angular speed

$$\Omega(r) - \Omega(r_\odot)$$

The speed of the cloud in this reference frame is

$$v = r \left[\Omega(r) - \Omega(r_{\odot}) \right]$$

(b) Use the diagram below to justify the various angular identifications to show that the radial velocity of the gas cloud is given by

$$v_r = r \left[\Omega(r) - \Omega(r_{\odot})\right] \sin\left(\theta + \ell\right)$$

where ℓ is the Galactic longitude of the cloud.



Solution: The direction of its velocity is tangent to the circle of radius r and thus perpendicular to the radius drawn from the Galactic center. We need to know the component of this velocity along the line of sight since this is the radial velocity v_r measured by the Doppler shift. We will begin by calculating the angles α , β , and γ in the diagram:



Solution: Next, apply the law of sines to the triangle defined by the Sun, the Galactic Center, and the cloud:

$$\frac{\sin\alpha}{r_{\odot}} = \frac{\sin\ell}{r}$$

However,

$$\sin \alpha = \sin \left(\pi - \theta - \ell \right) = \sin \left(\theta + \ell \right)$$

Substituting back into the law of sines expression and cross- multiplying gives

$$r\sin\left(\theta + \ell\right) = r_{\odot}\sin\ell$$

Plugging this back into the expression for v_r gives

$$v_r = r_{\odot} \left[\Omega(r) - \Omega(r_{\odot}) \right] \sin \ell$$

(d) For a given Galactic longitude ℓ , the only thing that can vary in Equation 1 is the galactocentric distance of the gas cloud. Argue that if $\Omega(r)$ is a monotonically decreasing function of r, then v_r acquires its maximum positive value (for $0^\circ \leq \ell \leq 90^\circ$) when r corresponds to the radius of the circular orbit tangent to the line of sight.

Solution: Now, by assumption, $\Omega(r)$ decreases monotonically with r (required for a flat rotation curve). Thus, the largest value of v_r in a given line of sight comes from the smallest r included in the line of sight, which in turn is the radius of the smallest circle intersected by the line of sight.

(e) Thus, show that the orbital speed at the tangent point (the speed we would want to use with Newton's laws or Kepler's laws, to work out masses) is given by

$$v(r) = r\Omega(r) = v_{r,\max} + r_{\odot}\Omega(r_{\odot})\sin\ell$$

where everything on the right hand side can be obtained from observations.

Solution: The line of sight is tangent to the smallest circle it intersects, so

$$\theta + \ell = \frac{\pi}{2}$$

for the cloud with the maximum radial velocity in this direction. According to the expression we derived above, this is

$$v_{r,\max} = r_{\min} \left[\Omega(r_{\min}) - \Omega(r_{\odot}) \right] \sin\left(\frac{\pi}{2}\right) = r_{\min} \left[\Omega(r_{\min}) - \Omega(r_{\odot}) \right]$$

where $r_{\min} = r_{\odot} \sin \ell$. For any line of sight we can write the circular velocity of clouds at the tangent point as

$$v(r_{\min}) = r_{\min}\Omega(r_{\min}) = v_{r,\max} + r_{\min}\Omega(r_{\odot}) = v_{r,\max} + r_{\odot}\Omega(r_{\odot})\sin\ell$$

The maximum radial velocity comes from the maximum Doppler shift in the spectrum. The Galactic longitude ℓ is the angle bretween the light of sight and the cloud. The Galactocentric distance r_{\odot} comes from proper motion measurements of objects near the Galactic center. And the angular speed in orbit comes from observations of stellar velocities in the Solar neighborhood and determination of the Oort constants A and B. Therefore, everything in the above expression is observable.

2. Galactocentric distance and the distance ambiguity: If we know r_{\odot} , $\Omega(r_{\odot})$, and the functional form of $\Omega(r)$, and we measure v_r , when we point a radio telescope in direction ℓ , the equation you derived in Problem 1 allows us to deduce a cloud's galactocentric distance r.

(a) Derive from this an expression for r. You can assume that the rotational tangential velocities in the disk are nearly constant with r.

Solution: Since $\Omega = v/r$, then

$$v_r = r_{\odot}[\Omega(r) - \Omega(r_{\odot})] \sin \ell$$
$$= r_{\odot} \left[\frac{v}{r} - \frac{v}{r_{\odot}} \right] \sin \ell$$
$$r = r_{\odot} \frac{v \sin \ell}{v_r + v \sin \ell}$$

Everything on the right-hand side of this expression is measurable. An observation of the radial Doppler velocity v_r and the longitude ℓ allows a determination of a cloud's galactocentric radius r.

(b) With a glance at the figure in Problem 1, show that if $r < r_{\odot}$ the line of sight generally intersects the circle of radius r at two points: a near point and a far point. This is the distance ambiguity. Show that the distance ambiguity does not arise if $r > r_{\odot}$.



The line of sight from the Sun intersects each inner orbit twice, for example at the points labeled 1 and 3.

Notice that the angle formed by the line of sight and the line from GC to 1 (defined by the triangle GC-1-2) is congruent to the angle between the line of sight and the line from GC to 3 (defined by the triangle GC-3-2). This angle, labeled $\theta + \ell$ in the figure from Problem 1, determines the component of the velocity v along the line of sight,

$$v_r = v\sin\left(\theta + \ell\right)$$

and so both points will have the same Doppler velocity v_r . Therefore we have no easy way to determine if the point is at location 1 or 3 unless we can observe the proper motion, which may not be practical — it takes millions of years for the objects at this radius to move appreciably with respect to our line of sight.

Thus, we have to use more indirect measurements to tell whether or not we're looking at point 1 or 3. For example, if the cloud has a visible-light counterpart it's probably at point 1, because

galactic dust would easily obscure a more distant object at point 3. Alternatively, if the angular size is large it's also more likely to be at point 1 than 3.

3. The azimuthally averaged surface brightness of the disk of a spiral galaxy is given in terms of distance R within the disk from the center of the galaxy by

$$\mathcal{L}(R) = \mathcal{L}(0)e^{-R/R_0}$$

Consider a nearly face-on spiral galaxy for which the "scale length" of the disk surface brightness is $R_0 = 2$ kpc, and for which the mass-to-light ratio is 2 at this galactocentric radius. Suppose you have shown that the rotation curve is flat over the galactocentric radii 2 - 12 kpc, so that you might suggest a spherical halo with density

$$\rho(r) = \rho_0 \left(\frac{r_0}{r}\right)^2$$

to dominate the mass in this range, where r is the usual spherical coordinate. Assume for simplicity that the core radius of the halo r_0 is equal to the scale length R_0 of the disk surface brightness.

(a) Show that the effective mass per unit area of the disk due to the spherical halo is

$$\mu(R) = \frac{\pi \rho_0 r_0^2}{R}$$

by integrating the density along the direction z perpendicular to the disk. *Hint*: Note that $r = \sqrt{R^2 + z^2}$ and use a trigonometric substitution to simplify the resulting integral.

Solution: First integrate the density along *z*:

$$\mu(R) = \int_{-\infty}^{\infty} \rho(R, z) \ dz = \rho_0 r_0^2 \int_{-\infty}^{\infty} \frac{dz}{R^2 + z^2} = \frac{\rho_0 r_0^2}{R} \int_{-\infty}^{\infty} \frac{dz/R}{1 + (z/R)^2}$$

Substitute $\tan \theta = z/R$, such that

$$\frac{z}{R} = \tan \theta$$
 $\frac{dz}{R} = \frac{d}{d\theta} \tan \theta = (1 + \tan^2 \theta) d\theta$

The range in this variable is

$$z \in (-\infty, \infty)$$
 $\theta = \arctan(z/R) \in (-\pi/2, \pi/2)$

Thus,

$$\mu(R) = \frac{\rho_0 r_0^2}{R} \int_{-\pi/2}^{\pi/2} \frac{(1 + \tan^2 \theta) d\theta}{1 + \tan^2 \theta} = \frac{\rho_0 r_0^2}{R} \int_{-\pi/2}^{\pi/2} d\theta = \frac{\pi \rho_0 r_0^2}{R}$$

(b) Plot (with Python or similar) the mass-to-light ratio $\mu(R)/\mathcal{L}(R)$ for R = 2 - 12 kpc. What is the value of the mass-to-light at the largest radii? How does this compare to the mass-to-light ratio in the solar neighborhood?

Solution: First we need to work out the ratio of the constant factors $\mathcal{L}(0)$ and ρ_0 . We have

$$\frac{M}{L}(R) = \frac{\mu(R)}{\mathcal{L}(R)} = \frac{\pi\rho_0 r_0^2/R}{\mathcal{L}(0)e^{-R/R_0}} = 2\frac{M_{\odot}}{L_{\odot}}$$
at $R = 2$ kpc, or
$$\frac{\rho_0}{\mathcal{L}(0)} = 2\left(\frac{2 \text{ kpc}}{\pi \cdot (2 \text{ kpc})^2}\right)e^{-1}\frac{M_{\odot}}{L_{\odot}} = \frac{1}{\pi e}\frac{M_{\odot}}{L_{\odot}} \text{ kpc}^{-1}$$
$$\frac{M}{L}(R) = \frac{r_0^2/R}{e^{-R/R_0+1}}\frac{M_{\odot}}{L_{\odot}} \text{ kpc}^{-1}$$

This is plotted below. Note that in the outer reaches of the galaxy the mass-to-light ratio is about $10 \times$ that in the Solar neighborhood. This must mean that the stellar mass function there is biased toward stars much less luminous than those near us, or that other forms of dark matter are present.



(c) By adjustment of the halo core radius r_0 , is it possible to make the mass-to-light ratio constant? Why or why not?

Solution: No. This would only work precisely if the mass and light distribution had the same functional form.

4. Cepheids in M31

(a) Suppose you found a classical Cepheid with a period of 30 days in M31, and you measure its average V magnitude to be 18.6. Calculate its absolute magnitude and estimate the distance to M31.

Solution: Use the relations given in lecture:

$$\overline{M_V} = -2.77 \log \Pi - 1.69 = -5.78$$
$$\overline{m_V} - \overline{M_V} = 5 \log (d/10 \text{ pc})$$
$$d = (10 \text{ pc}) 10^{(\overline{m_V} - \overline{M_V})/5}$$
$$= 0.752 \text{ Mpc}$$

(b) W Virginis stars (Population II Cepheids) are a factor of four less luminous than classical Cepheids for the same pulsation periods. By what factor would a derived distance to M31 be in error if M31 Cepheids of one type were mistakenly identified as the other? (This mistake was unwittingly made by Hubble before a distinction between the two types was discovered.)

Solution: The flux measured is the same in either case, but the above equations used to turn the flux and luminosity into a distance are equivalent to

$$f = \frac{L}{4\pi r^2} = \frac{L'}{4\pi r'^2},$$

where we can take L to be the Cepheid I luminosity and L' the W Virginis luminosity (which is a factor of 4 smaller). Thus, the ratio between the distance r' inferred under the assumption that the star is a Pop. II Cepheid and the distance r if it were a Pop. I Cepheid is

$$\frac{r'}{r} = \sqrt{\frac{L'}{L}} = \frac{1}{2},$$

i.e., the distance would be underestimated by a factor of 2.