1. Quasar luminosities

(a) Suppose that a quasar is as bright as a solar-type star (they share similar *apparent* magnitudes) but the quasar is a factor of a million further away than the star. What is the quasar's luminosity?

Solution: The luminosity ratio is

$$\frac{L'}{L} = \left(\frac{r'}{r}\right)^2$$

and taking $L = L_{\odot}$ and $r' = 10^6 r$, we get $L' = 10^{12} L_{\odot}$ for the quasar. That is quite a large luminosity — our Galaxy only puts out a total of about $2 \times 10^{10} L_{\odot}$.

(b) A certain quasar has an average luminosity of $3.3 \times 10^{13} L_{\odot}$ and an X-ray brightness that can vary substantially in as little as three hours. Assume that the quasar's black-hole engine is accreting at the Eddington rate and show that these two findings are consistent with each other.

Solution: If the central BH is producing luminosity at the Eddington limit, then its mass is

$$M = \frac{2e^4L}{3Gm_p m_e^2 c^5} = 2 \times 10^{42} \text{ g} = 10^9 M_{\odot}$$

Note that in CGS, the units of e are statcoulombs, or

$$e = 4.803 \times 10^{-10} \text{ cm}^{3/2} \text{ g}^{1/2} \text{ s}^{-1}$$

It is easiest to do the problem in these units.

The Schwarzschild radius of a $10^9 M_{\odot}$ black hole is

$$R_{\rm Sch} = \frac{2GM}{c^2} = 3 \times 10^{14} \text{ cm} = 3 \times 10^{-4} \text{ ly} = 2.8 \text{ light hours}$$

about the same as the "variablity size" of the X-ray emitting region. This is not shocking, since the X-rays would arise in the hottest part of the accretion disk, and that would be the bit closest to the event horizon.

- 2. Mass accretion can be used to power things besides active galaxies and can use engines besides black holes. A "junior" version of an AGN black hole is a $0.5M_{\odot}$ protostar accreting fully ionized hydrogen gas from its surrounding disk and producing luminosity from this accretion at the Eddington rate.
 - (a) What is its luminosity in L_{\odot} ?

Solution:

$$L = \frac{3GMm_pm_e^2c^5}{2e^4} = 6.3 \times 10^{37} \text{ erg/s} = 1.6 \times 10^4 L_{\odot}$$

(b) Suppose the accreted material falls freely to the star's surface from a distance much greater than the protostar radius of $1.5R_{\odot}$. At what rate in M_{\odot} yr⁻¹ is the protostar accreting matter?

Solution: If a mass m falls from infinity to radius $r = 1.5R_{\odot}$ from a mass M, its potential energy has changed by -GMm/r and since the infalling material is effectively brought to rest on the surface of the star, all of this energy is converted to heat. That the material is brought to rest at the stellar surface, rather than being in orbit, means that you cannot use the virial

theorem here. Therefore,

$$L = \frac{d}{dt} \left(\frac{GMm}{r}\right) = \frac{GM}{r} \frac{dm}{dt}$$
$$\frac{dm}{dt} = \frac{rL}{GM} = 1 \times 10^{23} \text{ g/s} = 1.5 \times 10^{-3} M_{\odot} \text{ yr}^{-1}$$

- 3. M87 has an accretion disk around its central black hole for which the rotational velocity has been measured in HST spectra. The disk extends 20 pc from the center and exhibits Doppler velocities as large as ± 500 km/s with respect to the galaxy's overall radial velocity.
 - (a) Calculate the mass of the black hole in M87 to two significant digits. Comment on the assumptions under which you did your calculation (e.g., orbital plane viewed edge-on; note the appearance of the galaxy and its disk in the notes) and the effect this may have on the accuracy of your answer.

Solution: If the orbit is Keplerian, the central mass can be obtained from the second law:

$$F = \frac{GMm}{r^2} = ma = \frac{mv^2}{r} \implies M = \frac{rv^2}{G}$$

If $v = v_r$ (i.e., the orbit is viewed edge-on), then this works out to be $1.16 \times 10^9 M_{\odot}$. Inspecting the HST and VLBI images, we see that the disk in M87 looks like it is inclined well away from edge on. It is hard to tell by how much, so we should regard the mass we worked out as a lower limit to the real mass. If we knew the angle θ between our line of sight and the plane of the disk, we would divide our edge-on answer by $\cos \theta$ to get the central mass.

(b) M87 is 16 Mpc away. With the mass you calculated for its central black hole, calculate the diameter of the black hole's event horizon (in pc) and the angle the event horizon subtends (in arcseconds).

Solution:

$$D_S = \frac{4GM}{c^2} = \frac{4(6.674 \times 10^{-8} \text{ dyn cm}^2 \text{ g}^{-2})(1.16 \times 10^9 M_{\odot})}{(3 \times 10^{10} \text{ cm/s})^2}$$
$$D_S = 6.86 \times 10^{14} \text{ cm} = 2.22 \times 10^{-4} \text{ pc}$$

$$\Delta \theta = \frac{D_S}{d} = \frac{2.22 \times 10^{-4} \text{ pc}}{16 \text{ Mpc}}$$

= 1.4 × 10⁻¹¹ rad = 2.9 × 10⁻⁶ arcsec

(c) Compare this result with the diffraction-limited angular resolution $\Delta \theta = 1.2\lambda/D$ (where D is telescope diameter) of the Hubble Space Telescope (D = 2.4 m) at a wavelength of 400 nm. Do the same for the VLBA (D = 8611 km) at a wavelength of 2 cm. Can we see details in images as small as the horizon? How far away would M87 have to be for the event horizon to subtend an angle equal to the best angular resolution?

Solution:

HST:
$$\Delta \theta = 1.2 \frac{\lambda}{D} = 2.0 \times 10^{-7} \text{ rad} = 0.041 \text{ arcsec}$$

VLBA: $\Delta \theta = 1.2 \frac{\lambda}{D} = 2.8 \times 10^{-9} \text{ rad} = 5.8 \times 10^{-4} \text{ arcsec}$

So neither HST nor VLBA has enough angular resolution to see details as small as the BH horizon of M87. We would have to move it closer:

$$\frac{4GM}{c^2 d} = 1.2 \frac{\lambda}{D}$$
$$d = \frac{GMD}{0.3c^2 \lambda} = \begin{cases} 79.7 \text{ kpc} & (\text{VLBA})\\ 1.11 \text{ kpc} & (\text{HST}) \end{cases}$$

The first of these distances would place M87 about twice as far away as the Large Magellanic Cloud. The second would put it inside our Galaxy.

- 4. The largest *apparent* superluminal motions seen so far in quasar jets are about 20c.
 - (a) What would this imply for the ejection speed v (in units of c to three significant figures) and jet angle with the line of sight θ (in degrees) if the jet is oriented for maximum apparent superluminal motion?

Solution: Just plug in and solve:

$$v_{\perp,\text{apparent}} = v \frac{1}{\sqrt{1 - v^2/c^2}} = 20c$$
$$= \frac{20c}{\sqrt{401}} = 0.999c$$
$$\theta = \arccos\left(\frac{v}{c}\right) = 2.86^{\circ}$$

(b) If this ejection speed applies to 3C 273, at what angle from the jet axis (in degrees) do we view this quasar?

Solution: We should expect two angles, one greater than and one less than 2.86°, because we are off the apparent superluminal motion maximum. 3C 273 has $v_{\perp,\text{apparent}} = 7c$, so for the given ejection speed of $v = 20c/\sqrt{401}$ the angles are determined by

$$v_{\perp,\text{apparent}} = \frac{v\sin\theta}{1 - v/c\cos\theta} = 7c$$

or, more simply,

$$\beta_{\perp} = \frac{\beta \sin \theta}{1 - \beta \cos \theta}$$

where $\beta = v/c = 20/\sqrt{401}$ and $\beta_{\perp} = v_{\perp,\text{apparent}}/c = 7$. We solve for θ by multiplying it out, squaring, eliminating sine and cosine terms using $\cos^2 \theta + \sin^2 \theta = 1$, and solving the resulting

quadratic equation. This could be solved with approximations since β is very close to 1 and $\beta_{\perp}^2 \approx \beta^2$, but we will solve exactly:

$$\beta_{\perp} - \beta_{\perp} \cos \theta = \beta \sin \theta$$
$$\beta_{\perp}^2 - \beta_{\perp}^2 \beta^2 \cos^2 \theta - 2\beta_{\perp}^2 \beta \cos \theta = \beta^2 \sin^2 \theta = \beta^2 - \beta^2 \cos^2 \theta$$
$$(\beta_{\perp}^2 + 1)\beta^2 \cos^2 \theta - 2\beta_{\perp}^2 \beta \cos \theta + (\beta_{\perp}^2 - \beta^2) = 0$$

Thus,

$$\begin{aligned} \cos\theta &= \frac{2\beta_{\perp}^{2}\beta \pm \sqrt{4\beta_{\perp}^{4}\beta^{2} - 4(\beta_{\perp}^{2} + 1)\beta^{2}(\beta_{\perp}^{2} - \beta^{2})}}{\sqrt{2(\beta_{\perp}^{2} + 1)\beta^{2}}} \\ &= \frac{\beta_{\perp}^{2} \pm \sqrt{\beta_{\perp}^{4} - (\beta_{\perp}^{2} + 1)(\beta_{\perp}^{2} - \beta^{2})}}{(\beta_{\perp}^{2} + 1)\beta} \\ &= \frac{\beta_{\perp}^{2} \pm \sqrt{\beta_{\perp}^{2}\beta^{2} + \beta^{2} - \beta_{\perp}^{2}}}{(\beta_{\perp}^{2} + 1)\beta} \\ &= \begin{cases} 0.999959 & \text{positive root} \\ 0.962489 & \text{negative root} \end{cases} \end{aligned}$$

for which $\theta = 0.52^{\circ}$ or 15.7° . The larger angle would of course lead to a larger projected length on the sky for a given physical length, and since the jet in 3C 273 is pretty long, $\theta = 15.7^{\circ}$ seems much more likely.