1. A certain quasar has an average luminosity of $3.3 \times 10^{13} L_\odot$ and an X-ray brightness that can vary substantially in as little as three hours. Assume that the quasar's black-hole engine is accreting at the Eddington rate and show that these two findings are consistent with each other.

Solution: If the central BH is producing luminosity at the Eddington limit, then its mass is

$$M = \frac{2e^4 L}{3Gm_p m_e^2 c^5} = 2 \times 10^{42} \text{ g} = 10^9 M_\odot$$

Note that in CGS, the units of e are statcoulombs, or

$$e = 4.803 \times 10^{-10} \text{ cm}^{3/2} \text{ g}^{1/2} \text{ s}^{-1}$$

It is easiest to do the problem in these units.

The Schwarzschild radius of a $10^9 M_\odot$ black hole is

$$R_{\text{Sch}} = \frac{2GM}{c^2} = 3 \times 10^{14} \text{ cm} = 3 \times 10^{-4} \text{ ly} = 2.8 \text{ light hours}$$

about the same as the "variablity size" of the X-ray emitting region. This is not shocking, since the X-rays would arise in the hottest part of the accretion disk, and that would be the bit closest to the event horizon.

2. M87 has an accretion disk around its central black hole for which the rotational velocity has been measured in HST spectra. The disk extends 20 pc from the center and exhibits Doppler velocities as large as ±500 km/s with respect to the galaxy's overall radial velocity.

   (a) Calculate the mass of the black hole in M87 to two significant digits. Comment on the assumptions under which you did your calculation (e.g., orbital plane viewed edge-on; note the appearance of the galaxy and its disk in the notes) and the effect this may have on the accuracy of your answer.

   Solution: If the orbit is Keplerian, the central mass can be obtained from the second law:

   $$F = \frac{GMm}{r^2} = ma = \frac{mv^2}{r} \qquad \Longrightarrow \qquad M = \frac{rv^2}{G}$$

   If $v = v_r$ (i.e., the orbit is viewed edge-on), then this works out to be $1.16 \times 10^9 M_\odot$.

   Inspecting the HST and VLBI images, we see that the disk in M87 looks like it is inclined well away from edge on. It is hard to tell by how much, so we should regard the mass we worked out as a lower limit to the real mass. If we knew the angle θ between our line of sight and the plane of the disk, we would divide our edge-on answer by $\cos\theta$ to get the central mass.

   (b) M87 is 16 Mpc away. With the mass you calculated for its central black hole, calculate the diameter of the black hole's event horizon (in pc) and the angle the event horizon subtends (in arcseconds).

   Solution:

   $$D_S = \frac{4GM}{c^2} = \frac{4(6.674 \times 10^{-8} \text{ dyn cm}^2 \text{ g}^{-2})(1.16 \times 10^9 M_\odot)}{(3 \times 10^{10} \text{ cm/s})^2}$$

   $$D_S = 6.86 \times 10^{14} \text{ cm} = 2.22 \times 10^{-4} \text{ pc}$$

$$\Delta\theta = \frac{D_S}{d} = \frac{2.22 \times 10^{-4}\ \text{pc}}{16\ \text{Mpc}}$$
$$= 1.4 \times 10^{-11}\ \text{rad} = 2.9 \times 10^{-6}\ \text{arcsec}$$

(c) Compare this result with the diffraction-limited angular resolution $\Delta\theta = 1.2\lambda/D$ (where D is telescope diameter) of the Hubble Space Telescope ($D = 2.4$ m) at a wavelength of 400 nm. Do the same for the VLBA ($D = 8611$ km) at a wavelength of 2 cm. Can we see details in images as small as the horizon? How far away would M87 have to be for the event horizon to subtend an angle equal to the best angular resolution?

Solution:

$$\text{HST:} \qquad \Delta\theta = 1.2\frac{\lambda}{D} = 2.0 \times 10^{-7}\ \text{rad} = 0.041\ \text{arcsec}$$
$$\text{VLBA:} \qquad \Delta\theta = 1.2\frac{\lambda}{D} = 2.8 \times 10^{-9}\ \text{rad} = 5.8 \times 10^{-4}\ \text{arcsec}$$

So neither HST nor VLBA has enough angular resolution to see details as small as the BH horizon of M87. We would have to move it closer:

$$\frac{4GM}{c^2 d} = 1.2\frac{\lambda}{D}$$
$$d = \frac{GMD}{0.3c^2\lambda} = \begin{cases} 79.7\ \text{kpc} & \text{(VLBA)} \\ 1.11\ \text{kpc} & \text{(HST)} \end{cases}$$

The first of these distances would place M87 about twice as far away as the Large Magellanic Cloud. The second would put it inside our Galaxy.

3. The largest *apparent* superluminal motions seen so far in quasar jets are about $20c$.

(a) What would this imply for the ejection speed v (in units of c to three significant figures) and jet angle with the line of sight θ (in degrees) if the jet is oriented for maximum apparent superluminal motion?

Solution: Just plug in and solve:

$$v_{\perp,\text{apparent}} = v\frac{1}{\sqrt{1 - v^2/c^2}} = 20c$$
$$= \frac{20c}{\sqrt{401}} = 0.999c$$
$$\theta = \arccos\left(\frac{v}{c}\right) = 2.86^\circ$$

(b) If this ejection speed applies to 3C 273, at what angle from the jet axis (in degrees) do we view this quasar?

Solution: We should expect two angles, one greater than and one less than 2.86°, because we are off the apparent superluminal motion maximum. 3C 273 has $v_{\perp,\text{apparent}} = 7c$, so for the given ejection speed of $v = 20c/\sqrt{401}$ the angles are determined by

$$v_{\perp,\text{apparent}} = \frac{v\sin\theta}{1 - v/c\cos\theta} = 7c$$

or, more simply,

$$\beta_\perp = \frac{\beta\sin\theta}{1-\beta\cos\theta}$$

where $\beta = v/c = 20/\sqrt{401}$ and $\beta_\perp = v_{\perp,\text{apparent}}/c = 7$. We solve for θ by multiplying it out, squaring, eliminating sine and cosine terms using $\cos^2\theta + \sin^2\theta = 1$, and solving the resulting quadratic equation. This could be solved with approximations since β is very close to 1 and $\beta_\perp^2 \approx \beta^2$, but we will solve exactly:

$$\begin{aligned}
\beta_\perp - \beta_\perp\cos\theta &= \beta\sin\theta \\
\beta_\perp^2 - \beta_\perp^2\beta^2\cos^2\theta - 2\beta_\perp^2\beta\cos\theta &= \beta^2\sin^2\theta = \beta^2 - \beta^2\cos^2\theta \\
(\beta_\perp^2+1)\beta^2\cos^2\theta - 2\beta_\perp^2\beta\cos\theta + (\beta_\perp^2 - \beta^2) &= 0
\end{aligned}$$

Thus,

$$\begin{aligned}
\cos\theta &= \frac{2\beta_\perp^2\beta \pm \sqrt{4\beta_\perp^4\beta^2 - 4(\beta_\perp^2+1)\beta^2(\beta_\perp^2-\beta^2)}}{\sqrt{2(\beta_\perp^2+1)\beta^2}} \\
&= \frac{\beta_\perp^2 \pm \sqrt{\beta_\perp^4 - (\beta_\perp^2+1)(\beta_\perp^2-\beta^2)}}{(\beta_\perp^2+1)\beta} \\
&= \frac{\beta_\perp^2 \pm \sqrt{\beta_\perp^2\beta^2 + \beta^2 - \beta_\perp^2}}{(\beta_\perp^2+1)\beta} \\
&= \begin{cases} 0.999959 & \text{positive root} \\ 0.962489 & \text{negative root} \end{cases}
\end{aligned}$$

for which $\theta = 0.52°$ or $15.7°$. The larger angle would of course lead to a larger projected length on the sky for a given physical length, and since the jet in 3C 273 is pretty long, $\theta = 15.7°$ seems much more likely.

4. Consider $10^{16}\ M_\odot$ of atomic hydrogen spread uniformly over a volume 10 Mpc in diameter and with a velocity dispersion $\sqrt{\overline{v^2}} = 1000$ km/s.

   (a) Take the hydrogen to be an ideal monatomic gas, so that the thermal energy per molecule is $E = 3k_BT/2$. What is the temperature of the big hydrogen cloud?

Solution:

$$\frac{1}{2}m_H\overline{v^2} = \frac{3}{2}k_BT \implies T = \frac{m_H\overline{v^2}}{3k_B} = 4\times10^7\ \text{K}$$

At this temperature the hydrogen would be ionized.

   (b) What is the Jeans mass of this material in $M_\odot$? What is the radius of a sphere with total mass equal to the Jeans mass (which we may as well call the **Jeans length**)?

Solution:
From the class notes, we have

$$M_J = \left(\frac{k_B T}{m_H G}\right)^{3/2} \left(\frac{3}{4\pi\rho}\right)^{1/2} \qquad R_J = \left(\frac{3M_J}{4\pi\rho}\right)^{1/3}$$

$$= \left(\frac{k_B T}{m_H G}\right)^{3/2} \left(\frac{R^3}{M}\right)^{1/2} \qquad = \left(\frac{M_J R^3}{M}\right)^{1/3}$$

$$= \boxed{7.6 \times 10^{13} M_\odot} \qquad = \boxed{0.98 \text{ Mpc}}$$

These values are quite big, i.e., much larger than the typical size of one galaxy.

(c) Compare your answer from part b to typical masses and sizes of galaxies and galaxy groups. If galaxy cluster-sized objects formed first in the early Universe, which formed next: galaxy-sized objects or galaxy group-sized objects?

Solution:
Assuming we start with a cloud the size of the Coma cluster, it would be expected to fragment into collapsing blobs of Jeans size (mass and radius). Those are much bigger than individual galaxies, as shown above, but are about the right size for galaxy groups.

Thus we would expect the cluster-sized clouds to fragment first into group-sized clumps, which then cool and fragment into galaxy-sized clumps. This is **hierarchical fragmentation**, in which large scales collapse first, followed by the small scales.

5. In a few billion years, our galaxy and the Andromeda galaxy will merge. Compute the expected number of collisions between stars when this occurs. Assume that the typical star in each galaxy is an M dwarf with a radius of $0.5R_\odot$, there are $N = 10^{11}$ stars in the Milky Way and 10^{12} stars in Andromeda, and that the average space density of stars in the Milky Way is $n = 1\ \text{pc}^{-3}$, equal to that in the solar neighborhood.

Solution:
The collisional cross section for each star is

$$\sigma = \pi R^2 = \frac{\pi[(0.5)(6.96 \times 10^8 \text{ m})]^2}{(3.1 \times 10^{16} \text{ m/pc})^2} = 4 \times 10^{-16} \text{ pc}^2$$

To determine the effective path through our galaxy, we want to compute the mean path length ℓ between collisions. First we note that

$$\ell \sim V^{1/3}$$

where V is the volume of stars in the galaxy. We can relate this to the number of stars in the Milky Way via the number density, which is just number per unit volume:

$$n = \frac{N}{V}$$

$$V = \frac{N}{n} = 10^{11} \text{ pc}^3$$

$$\ell \approx 4.6 \text{ pc}$$

Thus the probability of a collision of any particular star is

$$\begin{aligned} n\sigma\ell &= (1\ \mathrm{pc}^{-3})(4 \times 10^{-16}\ \mathrm{pc}^2)(4.6 \times 10^3\ \mathrm{pc}) \\ &= 1.8 \times 10^{-12} \end{aligned}$$

So with 10^{12} stars, the expected number of collisions is

$$(10^{12})(1.8 \times 10^{-12}) = 1.8$$

I.e., only a few collisions are even likely to occur.