1. The Lorentz transformation between two inertial reference frames with coordinate systems (x, t) and (x', t'), with the latter moving at constant speed v in the $+x$ direction, is

$$x' = \gamma(x - vt) \qquad y' = y \qquad z' = z \qquad t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Suppose two events are observed by experimenters in each reference frame. The intervals between their coordinates in the "unprimed" coordinate system are $\Delta x = x_1 - x_2$, $\Delta y = y_1 - y_2$, $\Delta z = z_1 - z_2$, and $\Delta t = t_1 - t_2$. Show that the intervals between the two events in the "primed" coordinate system have different values than in the unprimed system, but that both observers agree on the value of the absolute interval

$$\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = c^2\Delta t'^2 - \Delta x'^2 - \Delta y'^2 - \Delta z'^2$$

2. Light is emitted at $t = 0$ from an object at $r = 0$ and arrives later at a telescope located at $r = R(t)r_*$. Use the Robertson-Walker absolute interval,

$$ds^2 = c^2dt^2 - d\ell^2 = c^2dt^2 - R^2(t)\left(\frac{1}{1 - kr_*^2}dr_*^2 + r_*^2 d\theta^2 + r_*^2 \sin^2\theta d\phi^2\right)$$

to show that the proper distance ℓ traveled by the light is not equal to the coordinate distance r unless the Universe is flat.

3. Consider a flat, matter-dominated universe, i.e., one in which $\Lambda = 0$ and $k = 0$. Solve the Friedmann equation for this universe, obtaining a relation between the scale factor and time since the Big Bang, and an expression for the present age of the universe. Compare your results to those from the universes discussed in class.

4. (a) Solve the Friedmann equation to obtain the t–a relation for an empty universe.
   (b) At what redshift do the corresponding look-back times (that is, times before the present) of the flat matter-dominated universe of Problem 3 and the flat empty universe differ by 10%? That is, at what redshift would the Hubble diagram for a flat matter-dominated universe depart by 10% from a straight-line Hubble relation? (*Hint*: It is permitted to let Mathematica, Wolfram Alpha, etc. do the algebra here; consider solving graphically rather than algebraically.)

5. Our Universe is well described by $H_0 = 74.03$ km s^{-1} Mpc^{-1}, $\Omega_{M_0} = 0.31$, and $\Omega_{\Lambda_0} = 0.69$. The temperature of the cosmic microwave background indicates that decoupling took place at redshift $z = 1089.9$.
   (a) What is the age of the Universe with these parameters?
   (b) How long after the Big Bang did decoupling occur, according to these parameters?

