1. The Lorentz transformation between two inertial reference frames with coordinate systems (x, t) and (x', t'), with the latter moving at constant speed v in the $+x$ direction, is

$$x' = \gamma(x - vt) \qquad y' = y \qquad z' = z \qquad t' = \gamma\left(t - \frac{vx}{c^2}\right)$$
$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Suppose two events are observed by experimenters in each reference frame. The intervals between their coordinates in the "unprimed" coordinate system are $\Delta x = x_1 - x_2$, $\Delta y = y_1 - y_2$, $\Delta z = z_1 - z_2$, and $\Delta t = t_1 - t_2$. Show that the intervals between the two events in the "primed" coordinate system have different values than in the unprimed system, but that both observers agree on the value of the absolute interval

$$\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = c^2\Delta t'^2 - \Delta x'^2 - \Delta y'^2 - \Delta z'^2$$

Solution: Since the motion v is along the x direction, we can immediately note that

$$\Delta y^2 = \Delta y'^2 \qquad \Delta z^2 = \Delta z'^2$$

So the Lorentz transformation only applies to coordinates x and t:

$$\begin{aligned}
\Delta x' &= \gamma(x_1 - vt_1) - \gamma(x2 - vt_2) \\
&= \gamma(\Delta x - v\Delta t) \\
&\neq \Delta x \\
\Delta t' &= \gamma\left(t_1 - \frac{vx_1}{c^2}\right) - \gamma\left(t_2 - \frac{vx_2}{c^2}\right) \\
&= \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right) \\
&\neq \Delta t
\end{aligned}$$

However,

$$\begin{aligned}
c^2\Delta t'^2 - \Delta x'^2 &= c^2\gamma^2\left(\Delta t - \frac{v\Delta x}{c^2}\right)^2 - \gamma^2(\Delta x - v\Delta t)^2 \\
&= c^2\gamma^2\left(\Delta t^2 - \frac{2v\Delta x\Delta t}{c^2} + \frac{v^2\Delta x^2}{c^4}\right) - \gamma^2(\Delta x^2 - 2v\Delta x\Delta t + v^2\Delta t^2) \\
&= c^2\Delta t^2\gamma^2\left(1 - \frac{v^2}{c^2}\right) - \Delta x^2\gamma^2\left(1 - \frac{v^2}{c^2}\right) \\
&= c^2\Delta t^2 - \Delta x^2
\end{aligned}$$

and therefore

$$\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = c^2\Delta t'^2 - \Delta x'^2 - \Delta y'^2 - \Delta z'^2$$

2. Light is emitted at $t = 0$ from an object at $r = 0$ and arrives later at a telescope located at $r = R(t)r_*$. Use the Robertson-Walker absolute interval,

$$ds^2 = c^2dt^2 - d\ell^2 = c^2dt^2 - R^2(t)\left(\frac{1}{1 - kr_*^2}dr_*^2 + r_*^2d\theta^2 + r_*^2\sin^2\theta d\phi^2\right)$$

to show that the proper distance ℓ traveled by the light is not equal to the coordinate distance r unless the Universe is flat.

Solution: Light travels along trajectories such that $ds^2 = 0$. Thus, on one hand we have

$$ds^2 = c^2 dt^2 - d\ell^2 = 0$$
$$\ell = c\int_0^t dt' = ct$$

And on the other hand, noting that $d\theta = d\phi = 0$ for any particular direction, we have

$$ds^2 = c^2dt^2 - R^2(t)\left(\frac{1}{1-kr_*^2}dr_*^2\right) = 0$$

and

$$R(t)\int_0^{r_*}\frac{dr'_*}{\sqrt{1-kr_*'^2}} = c\int_0^t dt' = ct = \ell$$

Thus,

$$\ell = R(t)\int_0^{r_*}\frac{dr'_*}{\sqrt{1-kr_*'^2}} = \begin{cases} R(t)\int_0^{r_*} dr'_* = R(t)r_* = r & k=0 \\ R(t)\int_0^{r_*}\frac{dr'_*}{\sqrt{1-r_*'^2}} = R(t)\sin^{-1} r_* \neq r & k=+1 \\ R(t)\int_0^{r_*}\frac{dr'_*}{\sqrt{1+r_*'^2}} = R(t)\sinh^{-1} r_* \neq r & k=-1 \end{cases}$$

So only in the case of a flat universe ($k=0$) is $\ell = r$.

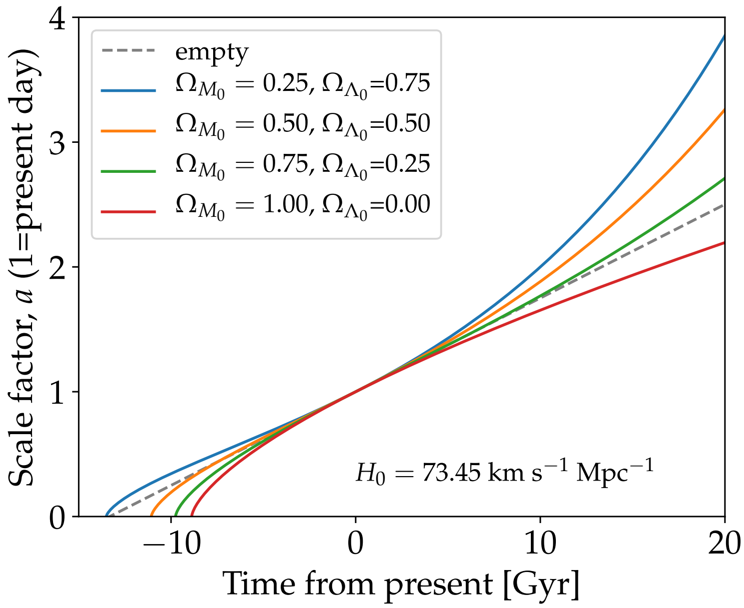
3. Consider a flat, matter-dominated universe, i.e., one in which $\Lambda = 0$ and $k = 0$. Solve the Friedmann equation for this universe, obtaining a relation between the scale factor and time since the Big Bang, and an expression for the present age of the universe. Compare your results to those from the universes discussed in class.

Solution:

In this universe $\Omega_{\Lambda_0} = 0$ and $\Omega_{M_0} = 1 - \Omega_{\Lambda_0} = 1$. Hence

$$\dot{a}^2 = H_0^2\left[1 + \left(\frac{1}{a} - 1\right)\right] = \frac{H_0^2}{a}$$
$$t = \frac{1}{H_0}\int_0^a \sqrt{a'}\, da'$$
$$= \frac{2a^{3/2}}{3H_0}$$

This solution is plotted, along with solutions for several other universes, in the lecture notes. It is plotted in red in the figure below:



Note that the form of $t(a)$ for the flat matter-dominated universe is the same as that to which *all* universes reduce for $a \ll 1$.

4. (a) Solve the Friedmann equation to obtain the t–a relation for an empty universe.

Solution:

$$\dot{a}^2 = H_0^2 \left[1 + \Omega_{M_0}\left(\frac{1}{a} - 1\right) + \Omega_{\Lambda_0}\left(a^2 - 1\right)\right] = H_0^2$$

$$t = \frac{a}{H_0}$$

(b) At what redshift do the corresponding look-back times (that is, times before the present) of the flat matter-dominated universe of Problem 3 and the flat empty universe differ by 10%? That is, at what redshift would the Hubble diagram for a flat matter-dominated universe depart by 10% from a straight-line Hubble relation? (*Hint*: It is permitted to let Mathematica, Wolfram Alpha, etc. do the algebra here; consider solving graphically rather than algebraically.)

Solution: Look-back time — let's call it τ — is time before the present. Since $a = 1$ at present,

$$\tau_f = t - t_0 = \frac{2a^{3/2}}{3H_0} - \frac{2}{3H_0} = \frac{2}{3H_0}(a^{3/2} - 1)$$

for flat matter-dominated universes, and

$$\tau_e = t - t_0 = \frac{a}{H_0} - \frac{1}{H_0} = \frac{1}{H_0}(a - 1)$$

for the empty universe. For a given a the empty universe curve lies further in the past than the flat matter-dominated one (see lecture). Thus, a positive 10% effect would be

$$\frac{\tau_e - \tau_f}{\tau_e} = 0.1 = \frac{\frac{1}{H_0}(a-1) - \frac{2}{3H_0}(a^{3/2}-1)}{\frac{1}{H_0}(a-1)}$$
$$0.1(a-1) = a - 1 - \frac{2}{3}(a^{3/2}-1)$$
$$a^{3/2} = \frac{27}{20}a - \frac{7}{20}$$

To find a you could plot $y = a^{3/2}$ and $20y = 27a - 7$ and read the value of a off from the intersection of the curves, or use a program such as Mathematica. Either way,

$$a = 0.627$$
$$z = \frac{1}{a} - 1 = 0.594$$

This is the typical Hubble-diagram departure which people expected to find as such redshifts when the search for deceleration began in the mid-1990s. As you know, they did find a departure but in the opposite direction (i.e., acceleration).

5. Our Universe is well described by $H_0 = 74.03$ km s^{-1} Mpc^{-1}, $\Omega_{M_0} = 0.31$, and $\Omega_{\Lambda_0} = 0.69$. The temperature of the cosmic microwave background indicates that decoupling took place at redshift $z = 1089.9$.

   (a) What is the age of the Universe with these parameters?

Solution: This is a flat universe, and so according to the lecture notes

$$t(a) = \frac{2}{3H_0\sqrt{1-\Omega}} \ln\left(\sqrt{\frac{1-\Omega}{\Omega}a^3} + \sqrt{1 + \frac{1-\Omega}{\Omega}a^3}\right), \text{ or}$$
$$t(z) = \frac{2}{3H_0\sqrt{1-\Omega}} \ln\left(\sqrt{\frac{1-\Omega}{\Omega}\left(\frac{1}{1+z}\right)^3} + \sqrt{1 + \frac{1-\Omega}{\Omega}\left(\frac{1}{1+z}\right)^3}\right)$$

where $\Omega = \Omega_{M_0}$. Thus, the age of the Universe is

$$t(z=0) = 12.63 \text{ Gyr}$$

   (b) How long after the Big Bang did decoupling occur, according to these parameters?

Solution:
If $z = 1089.9$ then

$$t(z = 1089.9) = 439.3 \text{ kyr}$$

The age of the Universe is a little shorter than that given by Planck and WMAP in their recent papers, while the decoupling time is a little late. The difference is almost entirely due to our neglect of the energy density of radiation in the Friedmann equation.