- 1. Consider  $10^{16} M_{\odot}$  of atomic hydrogen spread uniformly over a volume 10 Mpc in diameter and with a velocity dispersion  $\sqrt{v^2} = 1000 \text{ km/s}$ .
  - (a) Take the hydrogen to be an ideal monatomic gas, so that the thermal energy per molecule is  $E = \frac{3}{2}k_BT$ . What is the temperature of the big hydrogen cloud?

Solution:

$$\frac{1}{2}m_H\overline{v^2} = \frac{3}{2}k_BT \implies T = \frac{m_H\overline{v^2}}{3k_B} = 4 \times 10^7 \text{ K}$$

At this temperature the hydrogen would be ionized.

(b) What is the Jeans mass of this material in  $M_{\odot}$ ? What is the radius of a sphere with total mass equal to the Jeans mass (which we may as well call the **Jeans length**)?

## Solution:

From the class notes, we have

$$M_J = \left(\frac{k_B T}{m_H G}\right)^{3/2} \left(\frac{3}{4\pi\rho}\right)^{1/2} \qquad \qquad R_J = \left(\frac{3M_J}{4\pi\rho}\right)^{1/3}$$
$$= \left(\frac{k_B T}{m_H G}\right)^{3/2} \left(\frac{R^3}{M}\right)^{1/2} \qquad \qquad = \left(\frac{M_J R^3}{M}\right)^{1/3}$$
$$= \boxed{7.6 \times 10^{13} M_{\odot}} \qquad \qquad = \boxed{0.98 \text{ Mpc}}$$

These values are quite big, i.e., much larger than the typical size of one galaxy.

(c) Compare your answer from part b to typical masses and sizes of galaxies and galaxy groups. If galaxy cluster-sized objects formed first in the early Universe, which formed next: galaxy-sized objects or galaxy group-sized objects?

## Solution:

Assuming we start with a cloud the size of the Coma cluster, it would be expected to fragment into collapsing blobs of Jeans size (mass and radius). Those are much bigger than individual galaxies, as shown above, but are about the right size for galaxy groups.

Thus we would expect the cluster-sized clouds to fragment first into group-sized clumps, which then cool and fragment into galaxy-sized clumps. This is **hierarchical fragmentation**, in which large scales collapse first, followed by the small scales.

2. In a few billion years, our galaxy and the Andromeda galaxy will merge. Compute the expected number of collisions between stars when this occurs. Assume that the typical star in each galaxy is an M dwarf with a radius of  $0.5R_{\odot}$ , there are  $N = 10^{11}$  stars in the Milky Way and  $10^{12}$  stars in Andromeda, and that the average space density of stars in the Milky Way is  $n = 1 \text{ pc}^{-3}$ , equal to that in the solar neighborhood.

## Solution:

The collisional cross section for each star is

$$\sigma = \pi R^2 = \frac{\pi [(0.5)(6.96 \times 10^8 \text{ m})]^2}{(3.1 \times 10^{16} \text{ m/pc})^2} = 4 \times 10^{-16} \text{ pc}^2$$

To determine the effective path through our galaxy, we want to compute the mean path length  $\ell$  between collisions. First we note that

 $\ell \sim V^{1/3}$ 

where V is the volume of stars in the galaxy. We can relate this to the number of stars in the Milky Way via the number density, which is just number per unit volume:

$$n = \frac{N}{V}$$
$$V = \frac{N}{n} = 10^{11} \text{ pc}^{3}$$
$$\ell \approx 4.6 \text{ pc}$$

Thus the probability of a collision of any particular star is

$$n\sigma\ell = (1 \text{ pc}^{-3})(4 \times 10^{-16} \text{ pc}^2)(4.6 \times 10^3 \text{ pc})$$
  
=  $1.8 \times 10^{-12}$ 

So with  $10^{12}$  stars, the expected number of collisions is

$$(10^{12})(1.8 \times 10^{-12}) = 1.8$$

I.e., only a few collisions are even likely to occur.

3. The Lorentz transformation between two inertial reference frames with coordinate systems (x, t) and (x', t'), with the latter moving at constant speed v in the +x direction, is

$$x' = \gamma(x - vt) \qquad y' = y \qquad z' = z \qquad t' = \gamma\left(t - \frac{vx}{c^2}\right)$$
$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Suppose two events are observed by experimenters in each reference frame. The intervals between their coordinates in the "unprimed" coordinate system are  $\Delta x = x_1 - x_2$ ,  $\Delta y = y_1 - y_2$ ,  $\Delta z = z_1 - z_2$ , and  $\Delta t = t_1 - t_2$ . Show that the intervals between the two events in the "primed" coordinate system have different values than in the unprimed system, but that both observers agree on the value of the absolute interval

$$\Delta s^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = c^2 \Delta t'^2 - \Delta x'^2 - \Delta y'^2 - \Delta z'^2$$

**Solution:** Since the motion v is along the x direction, we can immediately note that

$$\Delta y^2 = \Delta y'^2 \qquad \qquad \Delta z^2 = \Delta z'^2$$

So the Lorentz transformation only applies to coordinates x and t:

$$\Delta x' = \gamma(x_1 - vt_1) - \gamma(x^2 - vt_2)$$
  
=  $\gamma(\Delta x - v\Delta t)$   
 $\neq \Delta x$   
$$\Delta t' = \gamma \left(t_1 - \frac{vx_1}{c^2}\right) - \gamma \left(t_2 - \frac{vx_2}{c^2}\right)$$
  
=  $\gamma \left(\Delta t - \frac{v\Delta x}{c^2}\right)$   
 $\neq \Delta t$ 

However,

$$c^{2}\Delta t'^{2} - \Delta x'^{2} = c^{2}\gamma^{2} \left(\Delta t - \frac{v\Delta x}{c^{2}}\right)^{2} - \gamma^{2}(\Delta x - v\Delta t)^{2}$$
$$= c^{2}\gamma^{2} \left(\Delta t^{2} - \frac{2v\Delta x\Delta t}{c^{2}} + \frac{v^{2}\Delta x^{2}}{c^{4}}\right) - \gamma^{2}(\Delta x^{2} - 2v\Delta x\Delta t + v^{2}\Delta t^{2})$$
$$= c^{2}\Delta t^{2}\gamma^{2} \left(1 - \frac{v^{2}}{c^{2}}\right) - \Delta x^{2}\gamma^{2} \left(1 - \frac{v^{2}}{c^{2}}\right)$$
$$= c^{2}\Delta t^{2} - \Delta x^{2}$$

and therefore

$$\Delta s^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = c^2 \Delta t'^2 - \Delta x'^2 - \Delta y'^2 - \Delta z'^2$$

4. Light is emitted at t = 0 from an object at r = 0 and arrives later at a telescope located at  $r = R(t)r_*$ . Use the Robertson-Walker absolute interval,

$$ds^{2} = c^{2}dt^{2} - d\ell^{2} = c^{2}dt^{2} - R^{2}(t)\left(\frac{1}{1 - kr_{*}^{2}}dr_{*}^{2} + r_{*}^{2}d\theta^{2} + r_{*}^{2}\sin^{2}\theta d\phi^{2}\right)$$

to show that the proper distance  $\ell$  traveled by the light is not equal to the coordinate distance r unless the Universe is flat.

**Solution:** Light travels along trajectories such that  $ds^2 = 0$ . Thus, on one hand we have

$$ds^{2} = c^{2}dt^{2} - d\ell^{2} = 0$$
$$\ell = c \int_{0}^{t} dt' = ct$$

And on the other hand, noting that  $d\theta = d\phi = 0$  for any particular direction, we have

$$ds^{2} = c^{2}dt^{2} - R^{2}(t)\left(\frac{1}{1 - kr_{*}^{2}}dr_{*}^{2}\right) = 0$$

and

$$R(t)\int_0^{r_*} \frac{dr'_*}{\sqrt{1-kr'^2_*}} = c\int_0^t dt' = ct = \ell$$

Thus,

$$\ell = R(t) \int_0^{r_*} \frac{dr'_*}{\sqrt{1 - kr'_*^2}} = \begin{cases} R(t) \int_0^{r_*} dr'_* = R(t)r_* = r & k = 0\\ R(t) \int_0^{r_*} \frac{dr'_*}{\sqrt{1 - r'_*}} = R(t)\sin^{-1}r_* \neq r & k = +1\\ R(t) \int_0^{r_*} \frac{dr'_*}{\sqrt{1 + r'_*}} = R(t)\sinh^{-1}r_* \neq r & k = -1 \end{cases}$$

So only in the case of a flat universe (k = 0) is  $\ell = r$ .