Introduction

Power series & approximations, Scaling relations, Characteristic scales, and Ensemble averages

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University of Rochester

Approximations in the equations of astrophysics

Astrophysical objects, be they planets, stars, nebulae, or galaxies, are all very complex compared to the physical systems you have so far encountered.

In order to simplify the relevant systems of equations that describe these objects to the point that they can be solved, astrophysicists employ approximations to the functions involved.

- The approximations used in introductory treatments of the subjects are often very crude, but they can still be useful in illuminating the general operating features of astrophysical systems.

Good, simple approximations can often be obtained from power-series representations of elementary functions.

Series expansions and first-order approximations

Power series expansions can be found using a Taylor Series expanded about $a = 0$:

$$f(a+x) = \sum_{m=0}^{\infty} f^m(x)\Big|_{x=a} \frac{x^m}{m!}$$

When $x \ll 1$, x^2, x^3, etc. are even smaller and have a negligible affect the Taylor series. A first-order expansion ignores terms of higher power than x^1.

What is the first-order approximation of $f(x) = \log_{10}(1+x)$?

$$f(x) \approx f(0) + \frac{f'(0)}{1!}x + \ldots = 0 + x\frac{df}{dx}\Big|_{x=0} + \ldots$$

To solve for $\frac{df}{dx}$, let $f(x) = y$. Then $e^{y \ln 10} = 1 + x$. Taking the derivative of both sides,

$$\frac{dy}{dx} = \frac{1}{(1+x)\ln 10}$$

So,

$$f(x) \approx \frac{x}{\ln 10}$$

Common expansions and first-order approximations

$$\sin x = \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \dots \approx x$$

$$\cos x = \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m}}{(2m)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots \approx 1$$

$$\tan x = \sum_{m=1}^{\infty} \frac{B_{2m}(-4)^m(1-4^m)}{(2m)!} x^{2m} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots \approx x$$

$$\arctan x = \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{2m+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \approx x$$

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2} + \dots \approx 1 + x$$

$$\ln(1+x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{m+1}}{m+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \approx x$$

$$(1+x)^n = \sum_{m=0}^{\infty} \frac{n!}{m!(n-m)!} x^m = 1 + nx + \frac{n(n-1)}{2} x + \dots \approx 1 + nx$$

First-order approximations

Find the approximations to first order in x.

$$\sqrt{e^x \cos x} \approx \sqrt{(1+x)1} \approx 1 + \frac{x}{2}$$

$$\frac{1}{e^x - 1} \approx \frac{1}{1 + x - 1} \approx \frac{1}{x}$$

$$\frac{4^n \tan x}{(2+x)^n (2-x)^n} = \frac{4^n \tan x}{(4-x^2)^n} \approx \frac{4^n \tan x}{4^n} \approx x$$

$$\frac{e^{ix} - e^{-ix}}{2i} \approx \frac{1 + ix - (1 - ix)}{2i} = x$$

Scaling relations

Sometimes the difference between results under different approximations or assumptions takes the form of a **common function of some key physical parameters**, multiplied by **different factors that are independent of these parameters**.

In this case, the cruder approximation gives a useful scaling relation.

Example

The mass density of a *uniform* sphere of mass M and radius R, at the center and throughout the sphere, is

$$\rho_0 = \frac{M}{V} = \frac{3}{4\pi}\frac{M}{R^3}$$

Scaling relations

Example

Solve for the mass density at the center of a spherically symmetric region of mass M and density that varies according to $\rho(r) = \rho_0 e^{-r/R}$.

$$\begin{aligned} M = \int_V \rho dV &= \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^{\infty} \rho(r) r^2 dr \\ &= 4\pi\rho_0 \int_0^{\infty} r^2 e^{-r/R} dr \\ &= 4\pi\rho_0 R^3 \int_0^{\infty} u^2 e^{-u} du \end{aligned}$$

The value of the integral is 2 (integrate by parts twice), so

$$M = 8\pi\rho_0 R^3 \qquad \therefore \rho_0 = \frac{1}{8\pi}\frac{M}{R^3}$$

Scaling relations

It appears as if the mass density at the center of the sphere will always have the form

$$\rho_0 = \begin{bmatrix}\text{factor independent of} \\ \text{mass and radius}\end{bmatrix} \times \frac{M}{R^3}$$

no matter what the details of the density. Common astrophysical nomenclature:

$$\rho_0 \propto \frac{M}{R^3}$$

- Jargon: The central density is said to be proportional to = "goes as" = "scales with" $\frac{M}{R^3}$

Why are scaling relations useful?

- Even if you do not know how the density varies with radius in the sphere, you would know what the ratio of central densities is for objects of a given size or mass ratio:
  - ρ_0 changes by a factor of $2^3 = 8$ if R changes by a factor of 2.
  - ρ_0 changes by a factor of 2 if M changes by a factor of 2.
- If we know everything about one "standard" object, scaling relations tell us a surprising amount about other similar objects for which we only know a few ratios of properties to the "standard."

Characteristic scales

Note that the sphere does not have a sharp edge in the exponential density case: $\rho(r) = \rho_0 e^{-r/R}$.

- R is not "the" radius of the sphere but rather a radius *characteristic* of the material in the sphere.
- Characteristic scales are important because they often suggest appropriate approximations.

Example

If we were making a calculation involving $\rho(r)$ and we were concerned about small r, we would mean "r small compared to R" and could apply the first-order approximation

$$\rho(r) = \rho_0 e^{-r/R} \approx \rho_0 \left(1 - \frac{r}{R}\right)$$

Scaling relations using known quantities

It is often convenient to peg a scaling relation to a **known quantity**. For example, the period of a planet orbiting a star of mass M at radius r is

$$P = 2\pi\sqrt{\frac{r^3}{GM}} \propto M^{-1/2} r^{3/2}$$

To use this, we can calculate $2\pi/\sqrt{G}$ every time, but it is better to re-express everything in terms of known time, mass, and distance units:

$$P = 1\ \text{year}\ \left(\frac{M}{M_\odot}\right)^{-1/2} \left(\frac{r}{1\ \text{AU}}\right)^{3/2}$$

This is convenient not only because we can use the scaling relation to understand how M and r affect P, but we also express it in units most convenient for our intuition.

Averages

Astrophysicists frequently cannot measure certain important parameters of a system, but may know how those parameters are **distributed** in a **population** of such systems.

- In this case we may use the average of the parameters.
- By "distribution of parameter x" we mean the probability $p(x)$ that x has a certain value, as a function of x.
- Convention: If p is a continuous function of x, and x can range over values from a to $b > a$, then it is normalized:

$$\int_a^b p(x)\,dx = 1 = 100\%$$

In other words, x must have some value between a and b.

Averages

Given the probability distribution $p(x)$, we define the average value of x as

$$\langle x \rangle = \int_a^b x\, p(x)\, dx$$

In general, the average value of any function of x, say $f(x)$, is similarly

$$\langle f(x) \rangle = \int_a^b f(x)\, p(x) dx$$

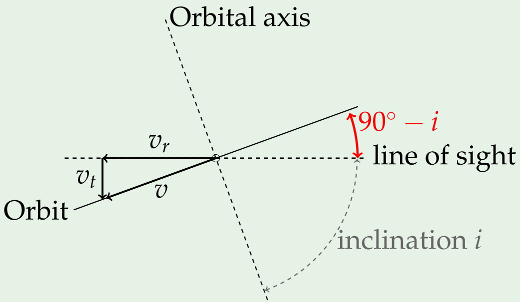
This is very similar to the sort of average you will learn in statistical mechanics, known as the ensemble average.

Averages: Binary star systems

Example

In binary star systems, astronomers can measure the component of orbital velocity along the line of sight for each star, i.e., the "radial velocity," v_r, via a Doppler shift.

- If the orientation of such a system was known — the inclination angle i of the system's axis with respect to our line of sight — the velocity measurements would allow us to derive the stars' masses.



$$v_r = v\cos\left(\frac{\pi}{2} - i\right) = v\sin i$$

Averages: Binary star systems

Example

- If the stars are too close to observe their orientation, then i cannot be determined.
- Still, an estimate of $\sin i$ would be useful. Could we make a reasonable assumption?
- Let us try an average value, assuming that binary star orbits are in general uniformly distributed. That is, all orientations are equally likely:

$$p(i) = \text{constant} = C, \quad i = 0 \to \frac{\pi}{2}$$

$$1 = \int_0^{\pi/2} p(i)di = C\int_0^{\pi/2} di = \frac{\pi C}{2}$$

$$\therefore C = \frac{2}{\pi}$$

Averages: Binary star systems

Example

Now that we know $p(i)$,

$$\langle \sin i \rangle = \int_0^{\pi/2} \sin i \, p(i) \, di = \frac{2}{\pi} \int_0^{\pi/2} \sin i \, di$$
$$= \left[-\frac{2}{\pi} \cos i \right]_0^{\pi/2} = -\frac{2}{\pi} (-1 - 0) = \frac{2}{\pi}$$

Thus, a reasonable *estimate* of the true orbital speed v given the *measured* line-of-sight speed v_r is

$$v = \frac{v_r}{\langle \sin i \rangle} = \frac{\pi}{2} v_r$$

Note that a more sophisticated treatment with *axes randomly oriented* yields $\langle \sin i \rangle = \pi/4$.

Averages: Binary star systems

Example

Allowing all *orientations* requires us to average over the full sky (Ω) defined by θ, ϕ:

$$p(\theta,\phi) = \text{constant} = C$$

$$1 = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta\, p(\theta,\phi) = 2\pi \cdot 2 \cdot C \qquad \therefore p(\theta,\phi) = \frac{1}{4\pi}$$

$$\langle \sin i \rangle = \int_\Omega d\Omega\, p(\theta,\phi) \sin\theta = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi} \sin^2\theta d\theta$$

$$= \frac{1}{2} \int_0^{\pi} \frac{1 - \cos 2\theta}{2} d\theta$$

$$= \frac{1}{4} \left[\theta - \frac{1}{2} \sin 2\theta \right]_0^{\pi} = \frac{\pi}{4}$$