Astronomy 142 — Recitation 3

Prof. Douglass

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Formulas to remember

Binary stars and their motions

All the ms and Ms here are masses; a is an orbital semimajor axis length; $a = r$ for circular orbits.

- Doppler effect
$$\frac{\lambda_{\text{observed}} - \lambda_{\text{rest}}}{\lambda_{\text{rest}}} = \frac{v_r}{c} \tag{1}$$
- Center of mass, circular orbits
$$\frac{m_1}{m_2} = \frac{r_2}{r_1} \tag{2}$$
- Kepler's third law, any eccentricity
$$P^2 = \frac{4\pi^2}{G(m_1 + m_2)}(a_1 + a_2)^3 \tag{3}$$
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- Conservation of momentum
$$\frac{m_1}{m_2} = \frac{a_2}{a_1} = \frac{v_{2r}}{v_{1r}} = \frac{v_2}{v_1} \tag{5}$$
- Mass function
$$f(m_1, m_2) = \frac{P}{2\pi G} v_{1r}^3 < m_2 \tag{6}$$

Hydrostatic equilibrium

Gravity v. pressure gradient — resulting in steady mass configuration

$$M(r) = \int_0^r \rho(r') 4\pi r'^2 \, dr'$$
$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$
$$P(R) - P(r) = \int_r^R \frac{dP}{dr'} dr' = -\int_r^R \frac{GM(r')\rho(r')}{r'^2} \, dr'$$
$$\text{Scaling: } \rho_C \propto \frac{M}{R^3}, P_C \propto \frac{GM^2}{R^4}, T_C \propto \frac{GM\mu}{R}$$

Ideal gases

$$P = \frac{\rho k T}{\mu} \tag{7}$$

where μ is the average mass of particles (i.e. atoms or ions or molecules) in the gas.

3-D random walk

To travel a distance R in randomly-directed steps of length ℓ requires going N steps, given by

$$N = \frac{3R^2}{\ell^2} \tag{8}$$

Workshop problems

Remember! The workshop problems you will do in groups in Recitation are a crucial part of the process of building up your command of the concepts important in ASTR 142 and subsequent courses. Do not, therefore, do your work on scratch paper and discard it. Better for each of you to keep your own account of each problem in some sort of bound notebook.

1. A star with total mass M has density given by

   $$\rho(r) = \rho_0 e^{-r/R} \tag{9}$$

   The star does not have a sharp outer edge; R is just the characteristic radius over which the density changes by a factor of $1/e$.

   What is the central density, ρ_0, in terms of M and the characteristic radius R?

2. *Stellar structure gets complicated quickly.* The following is a demonstration as to why we will consider stars to have a density with a functional form simpler than above for the remainder of the semester.

   (a) Using the results from above for a star with density

   $$\rho(r) = \rho_0 e^{-r/R}$$

   and the hydrostatic equilibrium equation, derive an expression for the pressure $P(r)$ at radius r within the star. You will need to use integration by parts and the integral shown below:

   $$\int_z^\infty \frac{e^{-x}}{x}\, dx \equiv Ei(z) \text{ the Exponential Integral)}$$[1]

   (b) Consider the behavior of this result at the center of the star. Can a star with this density be in hydrostatic equilibrium?

3. What is the total gravitational potential energy U for a *uniform*-density star with mass M and radius R, in terms of M and R?

Learn your way around the sky, lesson 3. (A feature *exclusive* of ASTR 142 recitations.) You may find the lab's celestial globes and the program Stellarium useful in answering these questions about the celestial sphere and the constellations.

[1] The Exponential Integral is an object that you will learn more about in Math 282. For real positive values of z, it is given by the following expression, obtainable by expanding the exponential integrated in a power series and integrating directly:

$$Ei(z) = \ln z + \sum_{j=1}^{\infty} \frac{z^j}{j \cdot j!} \tag{10}$$

4. *Sidereal time and hour angle*

   (a) A day is measured from noon to noon: the time it takes to return the Sun to its maximum elevation in the sky. Assume that Earth's orbit is circular. Calculate the angle η (in radians) that Earth's orbital radius sweeps out in a day.

   (b) By what angle, in radians, does Earth turn on its axis in the course of a day?

   (c) Earth spins at a constant rate. How long does it take, in hours, minutes, and seconds, for Earth to make one complete rotation with respect to the distant stars? This duration is called the *sidereal day*; it is the time it takes to return the fixed stars to their positions in the sky.

   Sidereal time tells us where the stars are in the sky, in the same way as ordinary civil time tells us where the Sun is in the sky. The rules are that the sidereal time is equal to the solar time at the moment of the vernal equinox: the time the Sun passes the celestial equator going north. The vernal equinox is also the origin of right ascension, so the sidereal time is also the right ascension of the stars that lie along the *meridian*: the line of celestial longitude connecting the poles and the *zenith* (the point directly overhead).

   (d) Suppose the vernal equinox happened precisely at noon on March 21. What is the right ascension of the star directly overhead in Rochester precisely at midnight on March 21? What is its declination?

   (e) During the same year as in part d: what is the sidereal time at midnight on September 21?

   (f) The coordinates of the black hole at the center of the Milky Way Galaxy, an object called Sagittarius A* (Sgr A*), are $\alpha = 17^{\mathrm{h}}45^{\mathrm{m}}40.0^{\mathrm{s}}$, $\delta = -29°00'28"$. On what date is Sgr A* on the meridian at midnight? (Or, on what date is the sidereal time at midnight equal to the object's right ascension?) Therefore, what would be a good time of year to observe Sgr A*?

Intro to Python, lesson 3. (A feature *exclusive* of ASTR 142 recitations.)

5. Consider a star of mass $M_* = M_\odot$ and radius $R_* = R_\odot$. Numerically integrate the equations of hydrostatic equilibrium and mass conservation,

$$\frac{dP}{dr} = -\frac{GM_r\rho}{r^2} \qquad \text{and} \qquad \frac{dM_r}{dr} = 4\pi r^2\rho \tag{12}$$

   to find $M(r)$ and $P(r)$ for three cases:

   (a) Uniform density: $\rho(r) = M_s/(4/3\ \pi R_s^3)$

   (b) Linear density: $\rho(r) = 3M_s/(\pi R_s^3)\,[1 - r/R_s]$

   (c) Quadratic density: $\rho(r) = 15M_s/(8\pi R_s^3)\,\left[1 - (r/R_s)^2\right]$

   Plot the mass and pressure for these cases as a function of r/R_s, and check that the value of the central pressure $P_{r=0} = P_c$ is sensible. (See `https://matplotlib.org/stable/gallery/index.html` for example plots and the corresponding code.)

   Hint 1: When numerically integrating, remember the boundary conditions $P_r \to 0$ as $r \to R_s$ and $M_r \to 0$ as $r \to 0$. This suggests that you may wish to integrate in opposite directions — surface to center and center to surface — to find M and P.

   Hint 2: There are many solvers available for systems of 1D ordinary differential equations, but you can approximate the integrals as simple Riemann sums

$$\int f(r)\ dr \approx \sum_i f(r_i)\Delta r_i \tag{13}$$

   for a suitably small choice of Δr_i.