The Sun

Stellar Structure
Structure of Sun's Outer Layers
Magnetism, Sunspots, and Flares
Energy and the Sun

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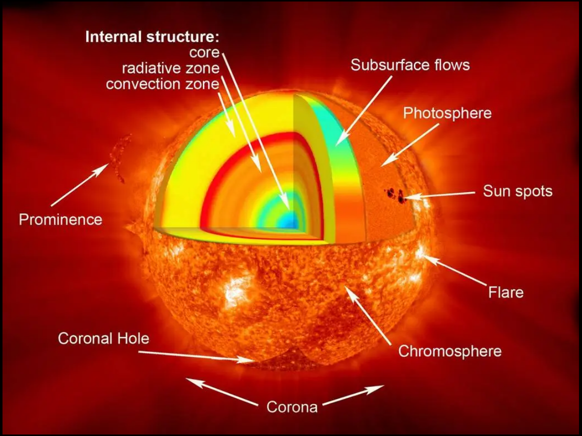
Stellar interiors & The Sun

- Principles of stellar structure
- Hydrostatic equilibrium
- Pressure, density, and temperature in the Sun's center
- Opacity of the Sun: diffusion of light from center to surface
- The structure of the Sun's outer layers: convection zone, photosphere, chromosphere, and corona
- Solar activity: magnetism, sunspots, and flares
- Solar energy



Layers of the Sun (image from NASA/Goddard).

Reading: Kutner Ch. 6, Ryden Sec. 7.1–7.2 & 15.1.3–15.2

Theoretical principles of stellar structure

- Vogt-Russell "Theorem:" **The mass and chemical composition of a star uniquely determine its radius, luminosity, internal structure, and subsequent evolution**. Not completely right but a very good approximation.
- Stars are **spherical** to a good approximation
- Stable stars are in hydrostatic equilibrium: the weight of each infinitesimal piece of the star's interior is balanced by the pressure differential across the piece.
  - Most of the time the pressure is *gas pressure* and is described well in terms of density and temperature by the ideal gas law

$$PV = NkT \qquad \text{or} \qquad P = \frac{\rho RT}{M}$$

  - However, in very hot or giant stars, the pressure exerted by light — *radiation pressure* — can dominate!

Theoretical principles of stellar structure

Energy is transported from the inside to the outside, most of the time in the form of **light**.

- The interiors of stars are **opaque**. Photons are absorbed and re-emitted many times on their way from the center to the surface in a random walk process called diffusion.
- The opacity depends on the density, temperature, and chemical composition.
- Most stars have regions in their interiors in which the radial variations of temperature and pressure are such that hot bubbles of gas can "boil" up toward the surface. This process, called **convection**, is a very efficient energy transport mechanism and can frequently be more important than light diffusion.

Basic equations of stellar structure

Hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{GM(r)\rho}{r^2}$$

Mass conservation

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho$$

Energy generation

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho \epsilon$$

Energy transport

$$\frac{dT}{dr} = -\frac{3}{16\sigma}\frac{\overline{\kappa}\rho}{r^2}\frac{L(r)}{4\pi r^2}$$

Adiabatic temperature gradient

$$\left(\frac{dT}{dr}\right)_{\mathrm{rad}} = -\left(1 - \frac{1}{\gamma}\right)\frac{\mu}{k}\frac{GM(r)}{r^2}$$

Convection occurs if

$$\frac{T}{P}\frac{dP}{dT} < \frac{\gamma}{\gamma - 1}$$

Equations of state

Pressure

$$P(\rho, T, \text{composition}) = \frac{\rho k T}{\mu} + \frac{4\sigma T^4}{3c}$$

in general throughout most normal stars.

Opacity

$$\overline{\kappa} = \overline{\kappa}(\rho, T, \text{composition}) \text{ in general}$$

Energy generation

$$\epsilon = \epsilon(\rho, T, \text{composition}) \text{ in general}$$

Boundary conditions are

$$\left.\begin{array}{ll} M(r) & \to 0 \\ L(r) & \to 0 \end{array}\right\} \text{as } r \to 0 \qquad \left.\begin{array}{ll} T & \to 0 \\ P & \to 0 \\ \rho & \to 0 \end{array}\right\} \text{as } r \to R_*$$

where $M(r)$ and $L(r)$ are the mass and luminosity contained within radius r.

Basic equations of stellar structure

This network of equations must be solved simultaneously, and there are no analytical solutions to the equations of stellar structure; usually, stellar-interior models are generated numerically. Does that mean we cannot do anything without a computer program?

- No; progress can be made by assuming a formula for one of the parameters and then solving for the rest.
- In this manner we can learn the workings of some of these differential equations and establish **scaling relations** useful in understanding the shapes of the empirical $R(M)$, $T_e(M)$, $L(M)$, and $L(T_e)$ results.
- It will be useful to first derive the stellar-structure equation we will use most: the equation of hydrostatic equilibrium.

Hydrostatic equilibrium

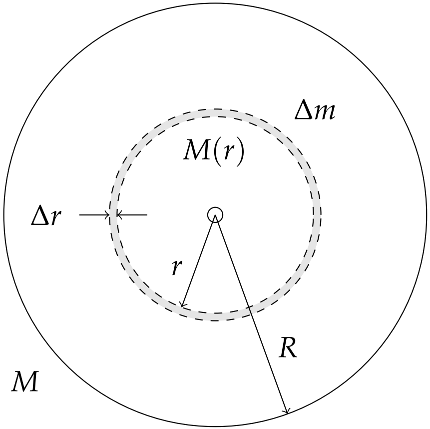
This is the principle that the gravitational force on any piece of the star is balanced by the pressure across the piece. This also holds for planetary atmospheres (see Kutner 23.3, Ryden 9.2, 14.1).

- Because planetary atmospheres are thin compared to the radii of planets, it is sufficient to approximate atmospheres as plane parallel slabs with constant gravitational acceleration. Thus,

  $$\frac{dP}{dz} = -\rho g$$

  a 1D Cartesian differential equation in z which can be solved in various cases.
- In stellar interiors, we still get a 1D differential equation (because stars are spherically symmetric), but we cannot ignore the spherical shape or the radial dependence of gravitational acceleration.

Hydrostatic equilibrium



$M(r)$: mass contained in radius r

Consider a spherical shell of radius r and thickness $\Delta r \ll r$ within a star with mass density (mass per unit volume) $\rho(r)$. Its weight is

$$\begin{aligned}\Delta F &= -\frac{GM(r)\Delta m}{r^2} \\ &= -\frac{GM(r)}{r^2}4\pi r^2 \Delta r \rho(r) \\ &= -4\pi GM(r)\rho(r)\Delta r\end{aligned}$$

Note the minus sign: force points inward.

Hydrostatic equilibrium

If the star is in hydrostatic equilibrium, then the weight is balanced by pressure differences across the shell:

$$\begin{aligned}
-\Delta F &= P(r)4\pi r^2 - P(r+\Delta r)4\pi(r+\Delta r)^2 \\
&\approx P(r)4\pi r^2 - \left[P(r) + \frac{dP}{dr}\Delta r\right]4\pi r^2 \\
&= -\frac{dP}{dr}4\pi r^2 \Delta r
\end{aligned}$$

to first order since $\Delta r \ll r$. Therefore,

$$\frac{dP}{dr} = \frac{\Delta F}{4\pi r^2 \Delta r} = -\frac{4\pi G M(r)\rho(r)\Delta r}{4\pi r^2 \Delta r} = \boxed{-\frac{GM(r)\rho(r)}{r^2}}$$

Crude stellar models

A surprisingly large amount about stellar interiors can be learned by starting with a formula for the mass density $\rho(r)$ and solving the equations of hydrostatic equilibrium and the equation of state self-consistently for pressure and temperature.

- The most useful is if the formula for density resembles the real thing; if not, P and T will be way off.
- We will consider problems with density in polynomial or exponential form in this course.
- Using simple expressions for the density, the hydrostatic equilibrium equation can be easily integrated.

Crudest approximation: the "uniform Sun"

We know the distance to the Sun (from radar measurements), its mass (from Earth's orbital period), and its radius (from angular size and known distance):

$$r = 1\ \text{AU} = 1.4960 \times 10^{13}\ \text{cm}$$
$$M_\odot = 1.988 \times 10^{33}\ \text{g}$$
$$R_\odot = 6.957 \times 10^{10}\ \text{cm}$$

Therefore, we know the average mass density (mass/volume) is

$$\rho_\odot = \frac{M_\odot}{V_\odot} = \frac{3M_\odot}{4\pi R_\odot^3} = 1.41\ \text{g/cm}^3$$
$$= 26\%\ \text{of Earth's density of} \sim 5.5\ \text{g/cm}^3$$

What would the pressure and temperature of the Sun be if it had *uniform density*?

The "uniform Sun"

With $\rho(r) = \bar{\rho}_\odot = 3M_\odot/4\pi R_\odot^3$,

$$\frac{dP}{dr} = -\frac{GM(r)\rho}{r^2} = -\frac{G}{r^2}\left(M_\odot \frac{r^3}{R_\odot^3}\right)\left(\frac{3M_\odot}{4\pi R_\odot^3}\right) = -\frac{3GM_\odot^2}{4\pi R_\odot^6} r$$

The surface has $P(R_\odot) = 0$ if it does not move, so

$$\int_{P(0)}^{0} dP = -\frac{3GM_\odot^2}{4\pi R_\odot^6}\int_0^{R_\odot} r\,dr$$

$$-P(0) = -\frac{3GM_\odot^2}{4\pi R_\odot^6}\frac{R_\odot^2}{2} \implies P(0) = \frac{3}{8\pi}\frac{GM_\odot^2}{R_\odot^4}$$

$$\therefore P_C = P(0) = 1.34 \times 10^{15}\ \mathrm{dyne/cm^2} = 1.3 \times 10^9\ \text{atmospheres}$$

The "uniform Sun"

Thus, the **pressure elsewhere** inside the uniform density Sun is

$$\int_{P_C}^{P(r)} dP' = -\frac{3GM_\odot^2}{4\pi R_\odot^6}\int_0^r r'\,dr' = P(r) - P_C = -\frac{3GM_\odot^2}{8\pi R_\odot^6} r^2$$

$$P(r) = \frac{3}{8\pi}\frac{GM_\odot^2}{R_\odot^4} - \frac{3}{8\pi}\frac{GM_\odot^2}{R_\odot^6} r^2$$

$$= \frac{3}{8\pi}\frac{GM_\odot^2}{R_\odot^4}\left(1 - \frac{r^2}{R_\odot^2}\right)$$

Now we can solve for the **temperature** using the fact that the Sun is an ideal gas:

$$PV = NkT \quad \Longrightarrow \quad P = nkT = \frac{\rho k T}{\mu}$$

where N is the number of particles, $n = N/V$ is number density, and μ is average particle mass.

The "uniform Sun"

Solving for temperature gives

$$T(r) = \frac{\mu P(r)}{k\rho} = \frac{\mu}{k}\frac{4\pi R_\odot^3}{3M_\odot}\frac{3}{8\pi}\frac{GM_\odot^2}{R_\odot^4}\left(1 - \frac{r^2}{R_\odot^2}\right)$$
$$= \frac{1}{2}\frac{\mu G M_\odot}{kR_\odot}\left(1 - \frac{r^2}{R_\odot^2}\right)$$

Suppose the average mass of the particles in this ideal gas is the same as the mass of the proton, $\mu = 1.67 \times 10^{-24}$ g. Then

$$T_C \equiv T(0) = 1.2 \times 10^7 \text{ K}$$

Note that in our approximations we end up with $T = 0$ at the surface, while in reality it is about 6000 K. However, that is indeed much less than 12 MK.

Central pressure in a star: scaling relation

Return for a moment to the central pressure in a uniform star:

$$P_C = \frac{3}{8\pi}\frac{GM^2}{R^4}$$

The only part of the equation which depends on the functional form of the density is the dimensionless coefficient $3/8\pi$. Otherwise the central pressure is proportional to M^2R^{-4}. As you will see in the homework, for densities such as

$$\rho(r) = \rho_C\left(1 - \frac{r}{R}\right) \quad \text{and} \quad \rho(r) = \rho_C\left[1 - \left(\frac{r}{R}\right)^2\right]$$

the central pressures are

$$P_C = \frac{5}{4\pi}\frac{GM^2}{R^4} \quad \text{and} \quad P_C = \frac{15}{16\pi}\frac{GM^2}{R^4}$$

Central pressure in a star: scaling relation

It turns out this is a simple scaling relation for stars:

$$P_C \propto \frac{GM^2}{R^4}$$

and the proportionality constant gets larger as the central density gets larger. A complete calculation for stars of low to moderate mass yields the expression

$$P_C \approx 19\frac{GM^2}{R^4}$$

For the Sun, $M_\odot = 1.99 \times 10^{33}$ g and $R_\odot = 6.96 \times 10^{10}$ cm, so

$$P_C \approx 19\frac{GM_\odot^2}{R_\odot^4} = 2.1 \times 10^{17} \text{ dyne cm}^{-2} > 10^{11} \text{ atmospheres}$$

Central pressure in a star: scaling relation

For other main sequence stars, we can derive an expression for the central pressure in terms of $M_\odot$ and $R_\odot$:

$$
\begin{aligned}
P_C &\approx 19\frac{GM^2}{R^4} = 19\frac{GM^2}{R^4}\left(\frac{M_\odot}{M_\odot}\right)^2\left(\frac{R_\odot}{R_\odot}\right)^4 \\
&= 19\frac{GM_\odot^2}{R_\odot^4}\left(\frac{M}{M_\odot}\right)^2\left(\frac{R_\odot}{R}\right)^4 \\
&= 19\frac{(6.67\times10^{-8}\ \mathrm{cm\ g^{-1}\ s^{-2}})(1.99\times10^{33}\ \mathrm{g})^2}{(6.96\times10^{10}\ \mathrm{cm})^4}\left(\frac{M}{M_\odot}\right)^2\left(\frac{R_\odot}{R}\right)^4 \\
&= 2.1\times10^{17}\left(\frac{M}{M_\odot}\right)^2\left(\frac{R_\odot}{R}\right)^4\ \mathrm{dyne/cm^2} \\
P_C &\approx 2\times10^{11}\left(\frac{M}{M_\odot}\right)^2\left(\frac{R}{R_\odot}\right)^{-4}\ \mathrm{atm}
\end{aligned}
$$

Central density and temperature of the Sun

The central pressure of the Sun is about 100 times larger than that of the "uniform" Sun. So what is the central density?

Guess: ρ_C is 100 times higher than the average density (equivalent to guessing that internal temperature does not vary much with radius).

- For the Sun, this is not a bad guess; the central density turns out to be 110 times the average density: $\rho_C = 150\ \mathrm{g/cm^3}$. So we can obtain another **scaling relation** from $\rho_C \propto MR^{-3}$:

$$\rho_C = 25\frac{M}{R^3} = 150\left(\frac{M}{M_\odot}\right)\left(\frac{R_\odot}{R}\right)^3\ \mathrm{g/cm^3}$$

Central density and temperature of the Sun

As we will see in a couple of weeks, the average gas-particle mass in the center of the Sun, considering its composition and the fact that the center is completely ionized, is $\mu_C = 1.5 \times 10^{-24}$ g. But the material is still an ideal gas, so

$$T_C = \frac{P_C V}{N_C k} = \frac{P_C}{n_C k} = \frac{P_C \mu_C}{\rho_C k} = 15.7 \times 10^6 \text{ K}$$

And indeed, T does not vary much with radius. We can make a **scaling relation** out of this as well to use on stars with different mass, radius, and composition:

$$T_C = \frac{P_C \mu_C}{\rho_C k} \propto \frac{GM^2}{R^4} \frac{R^3}{M} \mu_C = 15.7 \times 10^6 \text{ K for the Sun}$$

$$T_C = 15.7 \times 10^6 \left(\frac{M}{M_\odot}\right) \left(\frac{R_\odot}{R}\right) \left(\frac{\mu}{\mu_\odot}\right) \text{ K}$$

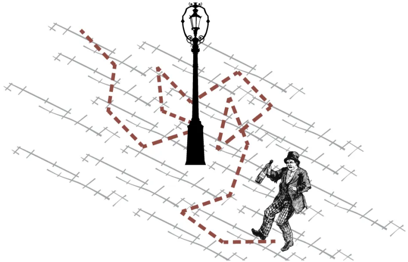
Opacity and luminosity in stars

At the high densities and temperatures found on average in stellar interiors, matter is opaque. The mean free path, or average distance a photon can travel before being absorbed, is about

$$\ell = 0.5 \text{ cm}$$

for the Sun's **average** density and temperature.

Photons produced in the center of the Sun have to randomly walk their way out, a process called diffusion.



Opacity and luminosity in stars

How many steps (in number of mean free paths) does it take for a photon to get from the center of the sun to the surface?

Let us work in 1D. Suppose a photon starts off at the center of the star and has an equal chance to go right or left after each absorption and re-emission step. The average position after N steps is

$$\langle x_N \rangle = (x_1 + x_2 + \cdots + x_N)/N = 0$$

However, the average value of the *square* of position is nonzero. Consider step $N+1$, assuming an equal chance of going left or right:

$$\begin{aligned} \langle x_{N+1}^2 \rangle &= \frac{1}{2}\langle (x_N - \ell)^2 \rangle + \frac{1}{2}\langle (x_N + \ell)^2 \rangle \\ &= \frac{1}{2}\langle x_N^2 - 2x_N\ell + \ell^2 \rangle + \frac{1}{2}\langle x_N^2 + 2x_N\ell + \ell^2 \rangle \\ &= \langle x_N^2 \rangle + \ell^2 \end{aligned}$$

Opacity and luminosity in stars

If this expression for $\langle x_{N+1}^2 \rangle$ is true for all N, then we can find $\langle x_N^2 \rangle$ by starting at zero and adding ℓ^2 N times (using induction):

$$\langle x_N^2 \rangle = N\ell^2$$

Thus, to randomly walk a distance $\sqrt{\langle x_N^2 \rangle} = L$ the photon needs to take, on average,

$$N = L^2/\ell^2 \text{ steps}$$

In 3D, the photon needs to take 3 times as many steps, so to travel a distance R it needs

$$N = \frac{3R^2}{\ell^2} \text{ steps}$$

Opacity and luminosity in stars

For the Sun, assuming a constant mean free path $\ell = 0.5$ cm and using $R = 6.96 \times 10^{10}$ cm,

$$N = \frac{3(6.96 \times 10^{10})^2}{(0.5)^2} = 5.81 \times 10^{22} \text{ steps}$$

This is very opaque!

Each step should take a time $\Delta t = \ell / c$, so the average time required for a photon to diffuse from the center of the Sun to the surface is

$$t = N\Delta t = \frac{3R_\odot^2}{\ell c} = 9.7 \times 10^{11} \text{ s} \approx 31,000 \text{ yr}$$

Note that the same trip takes only $t = R_\odot / c = 2.3$ s for a photon traveling in a straight line!