Astronomy 142 — Recitation #5

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Formulas to remember

Black holes

Gravitational time dilation and space warping A time interval $\Delta \tau$ and small (\ll coordinate radius r) radial interval $\Delta \mathcal{L}$ measured by a "local" observer near a black hole are related to Δt and Δr measured by a distant observer by

$$\Delta t = \frac{\Delta \tau}{\sqrt{1 - \frac{R_{\rm Sch}}{r}}} > \Delta \tau \qquad \Delta \mathcal{L} = \frac{\Delta r}{\sqrt{1 - \frac{R_{\rm Sch}}{r}}} > \Delta r \tag{1}$$

Note that $\Delta \mathcal{L}$ is measured by the local observer instantaneously, in the manner of a measuring tape; the positions of the ends of $\Delta \mathcal{L}$ are measured simultaneously by that observer. Note also that $\Delta \tau$ is a time interval between two events in the same location (like the ticks of a clock) in the reference frame of the local observer.

The main sequence (theoretically)

$$\ell \propto \begin{cases} \frac{T^{3.5}_{\overline{\rho}^2} & \text{if } M \le M_{\odot} \\ \frac{1}{\overline{\rho}} & \text{if } M \ge M_{\odot} \end{cases} \qquad \qquad L \propto \begin{cases} \frac{M^{5.5}_{\overline{R}^{0.5}} & \text{low mass} \\ M^3 & \text{high mass} \end{cases}$$
$$L = L_{\odot} \left(\frac{M}{M_{\odot}}\right)^4 \qquad \qquad R = R_{\odot} \left(\frac{M}{M_{\odot}}\right)$$
$$T_e = T_{e,\odot} \sqrt{\frac{M}{M_{\odot}}} \qquad \qquad L = L_{\odot} \left(\frac{T_e}{T_{e,\odot}}\right)^8$$

Mean molecular weight

$$\frac{m_p}{\mu} \cong 2X + \frac{3}{4}Y + \frac{1}{2}Z \tag{2}$$

where X, Y, and Z are the fractions of the star's mass in the form of hydrogen, helium, and all other elements, respectively.

Simulations

Cluster HR diagram evolution Animation of MESA stellar model results by Richard Pogge at Ohio State University. https://www.youtube.com/watch?v=wbvgjzW3Xz0

Workshop problems

Remember! The workshop problems you will do in groups in Recitation are a crucial part of the process of building up your command of the concepts important in ASTR 142 and subsequent courses. Do not, therefore, do your work on scratch paper and discard it. Better for each of you to keep your own account of each problem in some sort of bound notebook.

- (a) An observer far from a black hole observes a clock, identical to her own, that lies near a black hole. The black hole's clock appears to tick at half the rate that hers does (it has twice the tick period that hers does). What is the coordinate radius of the black hole's clock, in units of the Schwarzschild radius?
 - (b) The same observer can see yet another clock, far from any black holes, and receding from her at a high speed; this clock also appears to tick at half the rate as hers. How fast is this clock receding from her, in units of the speed of light?
 - (c) Observers at each of those clocks look back at the clock held by the original observer. By what factor, faster or slower, does the rate at which her clock ticks seem to differ from theirs?
- 2. A non-spinning black hole has horizon circumference equal to 2π in some units. A local observer sets particles into orbits, the circumferences of which are 4π , 6π , and 8π in the same units. The local observer compares the orbit circumferences he measures with those measured by a distant observer; the two sets agree.
 - (a) The distant observer measures the distances between the rings of particles using surveying techniques. What does she get?
 - (b) The local observer measures the distances between the rings by running a measuring tape in the radial direction between them. What does he get? (Note that these radial intervals are *not* small compared to the coordinate radii!)

Hint: The integral you need to do is not that difficult, but it does take more time than I would like you to spend on it, so I will give it to you:

$$\int \frac{dr}{\sqrt{1-\frac{1}{r}}} = \frac{1}{2} \ln \left(\frac{1}{\sqrt{1-\frac{1}{r}}} + 1 \right) - \frac{1}{2} \ln \left(\frac{1}{\sqrt{1-\frac{1}{r}}} - 1 \right) + \sqrt{r(r+1)} + C$$
(3)

- 3. Evolution on the main sequence for a fully convective star. A M3 main sequence star has mass $0.3M_{\odot}$, radius $0.3R_{\odot}$, and luminosity $0.036L_{\odot}$. Stars of low mass like this one turn out to be fully convective, so the elemental abundances are uniform throughout the star. For simplicity, suppose that the mass fractions of hydrogen (X) and helium (Y) start off at 0.75 and 0.25, respectively; neglect the heavier elements.
 - (a) How many helium nuclei are produced per second? What, therefore, are the rates of change of the hydrogen and helium mass fractions, in terms of the star's luminosity?
 - (b) If the star remains fully convective for its entire life, μ is uniform for its entire life; this means that no core of helium "ash" builds up, and thus there is no reason for the center to gradually heat up. It is safe to then assume that the central temperature stays constant. Obtain an expression for the radius of the star as a function of μ .
 - (c) What, then, will be the radius of the star if all the hydrogen is converted to helium? How does the star move in the HR diagram while this change is taking place?
- 4. A uranium star. The longest-lived isotope of element 92 has a mass number 238. (The atomic mass unit $u = 1.661 \times 10^{-24}$ g.) Suppose that it were possible to build a $1M_{\odot}$ star out of pure ²³⁸U.
 - (a) What is the average molecular mass, μ ?

- (b) What is the central temperature, presuming that the functional behavior of density and pressure are the same as in the Sun (i.e. $P_0 = 19GM^2/R^4$, etc.)?
- (c) Qualitatively describe the evolution of this star. What nuclear processes can it use to generate energy, and what effect do the nuclear processes have on the average molecular mass? (Note that ²³⁸U is not "fissile" like ²³⁵U.)
- 5. In an attempt to identify the most important components of mass loss from giant and supergiant stars, several researchers have postulated parameterizations of the mass loss rate that are based on fitting observed rates for a specified set of stars with some general equation that includes measurable quantities associated with the stars in the sample. One of the most popular equations for AGB stars is that by Reimers:

$$\frac{dM}{dt} = -\frac{\eta LR}{GM} \tag{9}$$

where L, M, and R are the star's luminosity, mass, and radius, respectively, and η is a dimensionless fitting parameter that turns out to be about 1.3×10^{-5} . The minus sign indicates that the star's mass is decreasing.

- (a) Explain qualitatively why L, M, and R enter this equation in the way that they do.
- (b) Estimate the mass-loss rate for a $1M_{\odot}$ AGB star that has luminosity $7000L_{\odot}$ and effective temperature 3000 K.
- (c) If the mass-loss rate stays constant at this value, how long does it take the star to be reduced to its degenerate carbon-oxygen core $(0.6M_{\odot})$?