Astronomy 142 — Recitation 5

Prof. Douglass

February 16, 2024

Formulas to remember

Pulsating stars

Adiabatic speed of sound:

$$v_s = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{\gamma k T}{\mu}} \tag{1}$$

$\gamma = 5/3$ for monatomic gases, $7/5$ for diatomic gases at room temperature.

Pulsation period, fundamental mode, uniform-density star:

$$\Pi = 4\int_0^R \frac{dr}{v_s} = \sqrt{\frac{6\pi}{\gamma G \rho}} \tag{2}$$

Special relativity

Lorentz factor

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \tag{3}$$

where v is the relative velocity between two inertial reference frames.

Time dilation

$$\Delta t = \gamma \Delta \tau \tag{4}$$

where $\Delta\tau$ is the "proper time" interval between two events in the same location (like the ticks of a clock) as measured by an observer at rest with respect to that location, and Δt is the time interval between those same two events as measured by an observer to whom the first observer seems to be moving at relative speed v.

Degenerate matter, white dwarfs, and neutron stars

Electron degeneracy pressure

$$P_e = \begin{cases} 0.0485\frac{h^2}{m_e}\left(\frac{Z}{A}\right)^{5/3}\left(\frac{\rho}{m_p}\right)^{5/3} & \text{(non-relativistic)} \\ 0.123 hc n_e^{4/3} & \text{(relativistic)} \end{cases} \tag{5}$$

Radius v. mass, non-relativistic limit

$$R = \begin{cases} 0.114\frac{h^2}{G m_e m_p^{5/3}}\left(\frac{Z}{A}\right)^{5/3} M^{-1/3} & \text{(electron degeneracy)} \\ 0.0685\frac{h^2}{G m_p^{8/3}} M^{-1/3} & \text{(neutron degeneracy)} \end{cases} \tag{6}$$

Central density and temperature, non-relativistic limit, white dwarf

$$P_C = 0.77\frac{GM^2}{R^4} \qquad \rho_C = 1.43\frac{M}{R^3} \tag{7}$$

Maximum mass of an electron-degeneracy-pressure-supported object

$$M_{SAC} = 0.2\left(\frac{Z}{A}\right)^2\left(\frac{hc}{Gm_p^2}\right)^{3/2} m_p = 1.44M_\odot \text{ for } \frac{Z}{A} = 0.5 \tag{8}$$

Black holes

Event horizon (Schwarzschild singularity) coordinate radius

$$R_{\text{Sch}} = \frac{2GM}{c^2} \tag{9}$$

The coordinate radius r is a distance measured by a distance observer from the center of the black hole.

Gravitational time dilation and space warping A time interval $\Delta\tau$ and small ($\ll$ coordinate radius r) radial interval $\Delta\mathcal{L}$ measured by a "local" observer near a black hole are related to Δt and Δr measured by a distant observer by

$$\Delta t = \frac{\Delta\tau}{\sqrt{1-\frac{R_{\text{Sch}}}{r}}} > \Delta\tau \qquad \Delta\mathcal{L} = \frac{\Delta r}{\sqrt{1-\frac{R_{\text{Sch}}}{r}}} > \Delta r \tag{10}$$

Note that $\Delta\mathcal{L}$ is measured by the local observer instantaneously, in the manner of a measuring tape; the positions of the ends of $\Delta\mathcal{L}$ are measured simultaneously by that observer. Note also that $\Delta\tau$ is a time interval between two events in the same location (like the ticks of a clock) in the reference frame of the local observer.

Workshop problems

Remember! The workshop problems you will do in groups in Recitation are a crucial part of the process of building up your command of the concepts important in ASTR 142 and subsequent courses. Do not, therefore, do your work on scratch paper and discard it. Better for each of you to keep your own account of each problem in some sort of bound notebook.

1. Newton worked on sound as well, but he turned out to be slightly off about how it propagated. He thought it was an isothermal pressure disturbance (constant temperature as pressure changes) instead of an adiabatic one (no heat flows from the pressure fluctuations). In modern terms, Newton's formula for the (isothermal) speed of sound would be

$$v_{s,isoT} = \sqrt{\frac{kT}{\mu}} \tag{11}$$

   where μ is the average mass of particles (atoms or molecules) in the gas. (Newton did not know about Boltzmann's constant or molecular masses, so he was unaware of what the proportionality constant should be in the relation he proposed for sound speed.)

   If Newton had been correct in these details, what would be the formula for pulsation periods of a uniform-density star, and by what factor do the periods for adiabatic and isothermal pulsations differ?

2. The quantum-mechanical, or de Broglie, wavelength corresponding to a particle with momentum $p_x = mv_x$ is $\lambda = \frac{h}{p_x}$.

(a) Present a brief argument to show that degeneracy pressure is important when a particle is confined to a space similar in size to, or smaller than, its de Broglie wavelength.

(b) Show from the relative sizes of their de Broglie wavelengths that electron degeneracy pressure can be larger than electron thermal pressure (so electrons are degenerate) while the nuclei that are mixed in with the electrons still behave like an ideal gas.

3. (a) You are a group of astronomers in the 1910s–1920s who have a lot of time on the Mt. Wilson 60- and 100-inch telescopes, and access to archival images going back about 50 years. From your archives and measurements, you find that Sirius A has a distance of 2.64 parsecs, that its orbit with Sirius B has a semimajor axis of angular length 7.56 arcsec and a period of 49.9 years, and that the semimajor axis of its own orbit (about the center of mass) is 2.41 arcsec. What are the masses of Sirius A and B, in units of the solar mass?

   (b) You figure out when Sirius B is at apastron, and take a spectrum of it then; the shape of its continuum indicates a temperature of 25,200 K, and the integral across the spectrum indicates a luminosity of $0.026L_\odot$ at the distance of 2.64 parsecs. What is the radius of Sirius B, in units of the solar radius?

4. (a) An observer far from a black hole observes a clock, identical to her own, that lies near a black hole. The black hole's clock appears to tick at half the rate that hers does (it has twice the tick period that hers does). What is the coordinate radius of the black hole's clock, in units of the Schwarzschild radius?

   (b) The same observer can see yet another clock, far from any black holes, and receding from her at a high speed; this clock also appears to tick at half the rate as hers. How fast is this clock receding from her, in units of the speed of light?

   (c) Observers at each of those clocks look back at the clock held by the original observer. By what factor, faster or slower, does the rate at which her clock ticks seem to differ from theirs?

5. A non-spinning black hole has horizon circumference equal to 2π in some units. A local observer sets particles into orbits, the circumferences of which are 4π, 6π, and 8π in the same units. The local observer compares the orbit circumferences he measures with those measured by a distant observer; the two sets agree.

   (a) The distant observer measures the distances between the rings of particles using surveying techniques. What does she get?

   (b) The local observer measures the distances between the rings by running a measuring tape in the radial direction between them. What does he get? (Note that these radial intervals are *not* small compared to the coordinate radii!)

   Hint: The integral you need to do is not that difficult, but it does take more time than I would like you to spend on it, so I will give it to you:

$$\int \frac{dr}{\sqrt{1-\frac{1}{r}}} = \frac{1}{2}\ln\left(\frac{1}{\sqrt{1-\frac{1}{r}}}+1\right) - \frac{1}{2}\ln\left(\frac{1}{\sqrt{1-\frac{1}{r}}}-1\right) + \sqrt{r(r+1)} + C \tag{13}$$

Learn your way around the sky, lesson 5. (A feature *exclusive* of ASTR 142 recitations.) You may find the lab's celestial globes and the program Stellarium useful in answering these questions about the celestial sphere and the constellations.

6. In Recitation #1, you considered a star with right ascension and declination α and δ, drew a Cartesian coordinate system with the x-axis pointing through $\alpha = \delta = 0$ and the z-axis pointing toward $\delta = 90°$, and derived an expression for a unit vector pointing at the star, in terms of the Cartesian unit vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$. You got

$$\hat{n} = \cos\delta\cos\alpha\hat{x} + \cos\delta\sin\alpha\hat{y} + \sin\delta\hat{z} \tag{14}$$

(a) In the same coordinate system, construct a unit vector that points toward the zenith and depends upon LST and latitude.

(b) Derive an expression for the angle between this new unit vector and the old one above. Look for a simplification that allows you to express the result in terms of the difference between LST and the right ascension of the object. This new angle is called the **zenith angle** (ZA); the right-angle complement, the altitude, is the angular distance between the star and the horizon. The difference between LST and RA is called the **hour angle** (HA).

Intro to Python, lesson 5. (A feature *exclusive* of ASTR 142 recitations.)

7. Show the relationship between a neutron star's mass and its size, average density, gravitational acceleration at the surface, and the rate of change of that acceleration, dg/dR.