Inside Stars

Nuclear fusion reactors in stars
Nucleosynthesis & the cosmic abundances of elements
Solar neutrinos
Pulsations in stars & the instability strip
Helioseismology
The standard solar model

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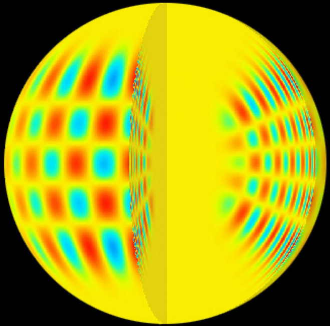
Seeing inside stars

Light does not penetrate stars, but neutrinos and acoustic waves (sound) do.

- Nuclear fusion reactors in stars
- Temperature dependence of the fusion rate in stars
- Nucleosynthesis and the cosmic abundances of the elements
- Solar neutrinos and the former solar neutrino problem
- Radial and nonradial pulsations in stars
- Pulsating stars and the instability strip
- Helioseismology
- The standard solar model

Reading: Kutner Ch. 9.5–9.6, Ryden Sec. 15.4 and 17.3



Pulsation with $n, \ell, m = 14, 20, 16$ *(Kosovichev et al. 1997).*

Fundamental physics: Matter

Matter is made of **quarks** and **leptons**. They come in three generations with two types of particle per generation:

Family	Generations			Notes
Quarks	u	c	t	Participate in all 4 interactions
	d	s	b	(strong, weak, E&M, gravity)
Leptons	e^-	μ^-	τ^-	Abstain from strong interaction
	ν_e	ν_μ	ν_τ	

All of these particles have spin-$\frac{1}{2}$ (i.e., $\frac{\hbar}{2}$). Thus:

- They are all **fermions**.
- They obey the Pauli exclusion principle.

Note: Each particle also has a corresponding antiparticle with some reversed quantum numbers. More on that in a moment.

Fundamental physics: Quarks

- Quarks have fractional electric charge of $\pm\frac{1}{3}e$, where e is the electron charge.
- They come in three different kinds of the strong-interaction analog of charge (**color**): red, green, and blue.
- Individual quarks are never observed (“confinement”).
- Nuclear particles are made of 2 or 3 quarks and are always *color-neutral*.
  - **Mesons**: quark-antiquark pairs (e.g., $\pi^{\pm}$, π^{0})
  - **Baryons**: three quarks (e.g., protons, neutrons)
- More exotic 4 and 5 quark states have also been created in accelerators (Swanson 2013, Aaij et al. 2014, Aaij et al. 2015).

Quantities conserved in the four interactions

- Energy, linear momentum, angular momentum, etc.
- Electric charge
- Lepton number, separately for each generation of leptons

Particles	ℓ_e	ℓ_μ	ℓ_τ
e^-, ν_e	1	0	0
μ^-, ν_μ	0	1	0
τ^-, ν_τ	0	0	1

- Baryon number. For example:

Particle	**Spin ($\hbar$)**	**Charge (e)**	**Baryon #**
u	$1/2$	$+2/3$	$1/3$
d	$1/2$	$-1/3$	$1/3$
proton (uud)	$1/2$	1	1
neutron (udd)	$1/2$	0	1

Antiparticles

Each of the quarks and leptons has a corresponding **antiparticle**.

- Antiparticles have exactly the same mass and spin as the corresponding particle.
- They have the **opposite** electric charge, lepton number, and baryon number of the corresponding particle.

Notation used in this class (not standard):

$$^{\text{baryon \#}}_{\text{charge}}\text{Particle} \qquad ^{\text{baryon \#}}_{\text{charge}}\overline{\text{Antiparticle}}$$

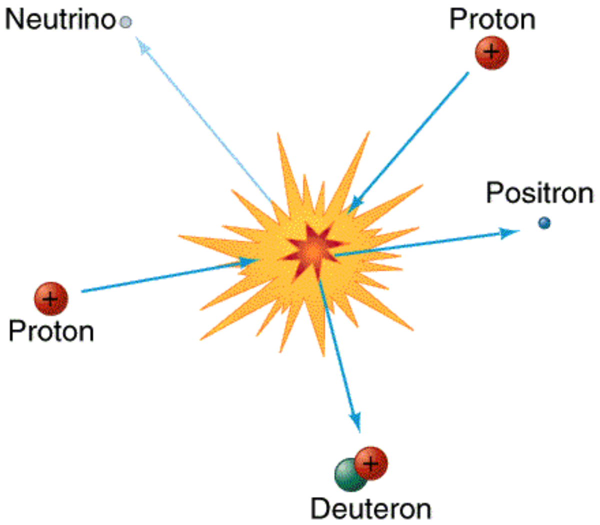
Examples:

$$\text{proton} = {}^1_1p \text{ or } {}^1_1\text{H} \qquad \text{antiproton} = {}^{-1}_{-1}\overline{p} \text{ or } {}^{-1}_{-1}\overline{\text{H}}$$

$$\text{electron} = {}^0_{-1}e \qquad \text{positron} = {}^0_1e^+$$

$$\text{photon} = {}^0_0\gamma \qquad \text{electron antineutrino} = {}^0_0\overline{\nu}_e$$

Fusion of two protons: $p + p \to {}^2\text{H} + e^+ + \nu_e$



From Chaisson & McMillan, Astronomy Today

Fusion and $E = mc^2$

Mass is a form of energy. Even when a body is at rest and is far from other attracting or repelling bodies, it has a total **rest energy**

$$E = m_0 c^2$$

where m_0 is the rest mass. In pp fusion:

- Two protons fuse to make a deuteron and two lightweight particles.
- The deuteron has a proton and neutron, both of which are about the same mass, but also a large negative potential energy from the strong interaction between them.
- Thus, the rest energy of the deuteron is *less* than the sum of the rest energies of the two protons. Equivalently, its rest mass is less than that of the protons.
- This suggests a convenient method for accounting for the energy released in fusion processes…

The proton-proton chains

Several different sequences of reactions starting with the fusion of two protons drive nuclear fusion in main sequence stars. Collectively, they are called the **proton-proton chains**.

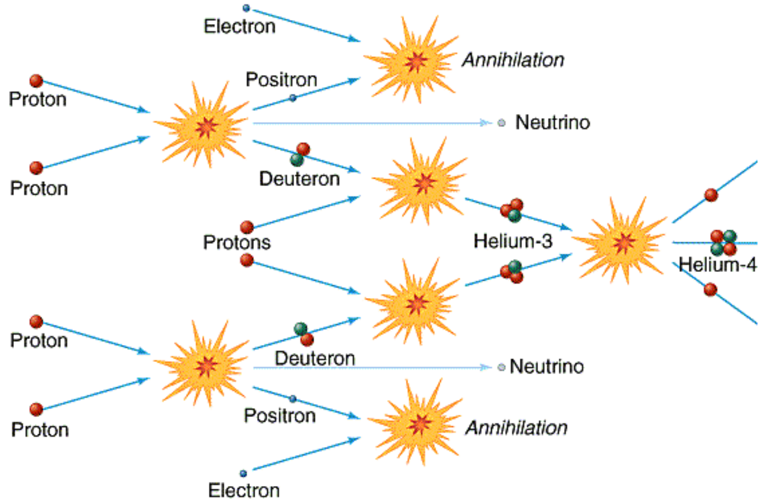
- Here is pp chain I (70% of pp chain reactions):

$$\begin{aligned} 2\,{}^{1}_{1}\mathrm{H} &\rightarrow {}^{2}_{1}\mathrm{H} + {}^{0}_{1}e^{+} + {}^{0}_{0}\nu_e && (\times 2) \\ {}^{2}_{1}\mathrm{H} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{3}_{2}\mathrm{He} + {}^{0}_{0}\gamma && (\times 2) \\ 2\,{}^{3}_{2}\mathrm{He} &\rightarrow {}^{4}_{2}\mathrm{He} + 2\,{}^{1}_{1}\mathrm{H} \end{aligned}$$

$$\textbf{Total: } 4\,{}^{1}_{1}\mathrm{H} \rightarrow {}^{4}_{2}\mathrm{He} + 2\,{}^{0}_{1}e^{+} + 2\,{}^{0}_{0}\nu_e + 2\,{}^{0}_{0}\gamma$$

- The rest mass of the products is less than the reactants, so the products have more kinetic energy than the reactants (heat!).

Proton-proton chain I (PPI)



From Chaisson and McMillan, Astronomy Today

The proton-proton chains

Note the application of all conservation laws:

$2\,{}^1_1\mathrm{H} \rightarrow {}^2_1\mathrm{H} + {}^0_1 e^+ + {}^0_0 \nu_e$

Mass and baryon number are conserved in the ${}^2_1$H. The extra charge (+) is carried off by a non-baryon, the positron (an anti-lepton). An electron neutrino is needed to conserve lepton number.

${}^2_1\mathrm{H} + {}^1_1\mathrm{H} \rightarrow {}^3_2\mathrm{He} + {}^0_0\gamma$

Energy and momentum cannot both be conserved unless there is more than one particle in the final state. A neutrino/antineutrino pair would also work here but would happen much less frequently.

The proton-proton chains

How much kinetic energy do the products have? The difference in rest mass between products and reactants gives the difference in binding (potential) energy:

$$\begin{aligned}\Delta W &= m_0(\mathrm{He})c^2 + 2m_0(e)c^2 - 4m_0(\mathrm{H})c^2 \\ &\approx m_0(\mathrm{He})c^2 - 4m_0(\mathrm{H})c^2 && \left(\text{Note: } \frac{m_p}{m_e} = 1836\right) \\ &= 3.97m_pc^2 - 4m_pc^2 && (\text{mass of He nucleus}) \\ &= -0.03m_pc^2 \\ &= \boxed{-4.5\times10^{-5}\ \mathrm{erg}}\end{aligned}$$

Compare to the average kinetic energy of particles in an ideal gas at 1.57×10^7 K:

$$\langle KE\rangle = \frac{3}{2}kT = 3\times10^{-9}\ \mathrm{erg}$$

Nuclear catalysis: the CNO bi-cycle

One branch, the CN cycle:

$$
\begin{aligned}
{}^{12}_{6}\mathrm{C} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{13}_{7}\mathrm{N} + {}^{0}_{0}\gamma \\
{}^{13}_{7}\mathrm{N} &\rightarrow {}^{13}_{6}\mathrm{C} + {}^{0}_{1}e^{+} + {}^{0}_{0}\nu_e \\
{}^{13}_{6}\mathrm{C} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{14}_{7}\mathrm{N} + {}^{0}_{0}\gamma \\
{}^{14}_{7}\mathrm{N} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{15}_{8}\mathrm{O} + {}^{0}_{0}\gamma \\
{}^{15}_{8}\mathrm{O} &\rightarrow {}^{15}_{7}\mathrm{N} + {}^{0}_{1}e^{+} + {}^{0}_{0}\nu_e \\
{}^{15}_{7}\mathrm{N} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{12}_{6}\mathrm{C} + {}^{4}_{2}\mathrm{He}
\end{aligned}
$$

$$\text{Total: } 4\,{}^{1}_{1}\mathrm{H} \rightarrow {}^{4}_{2}\mathrm{He} + 2\,{}^{0}_{1}e^{+} + 2\,{}^{0}_{0}\nu_e + 3\,{}^{0}_{0}\gamma$$

${}^{12}_{6}\mathrm{C}$ is a **catalyst**; it is *not* used up in the reactions.

This has the same rest mass difference, and therefore kinetic energy in the products, as the pp chains. It requires higher T, though.

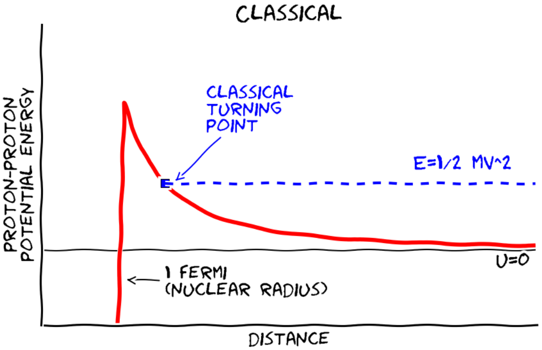
Hot fusion

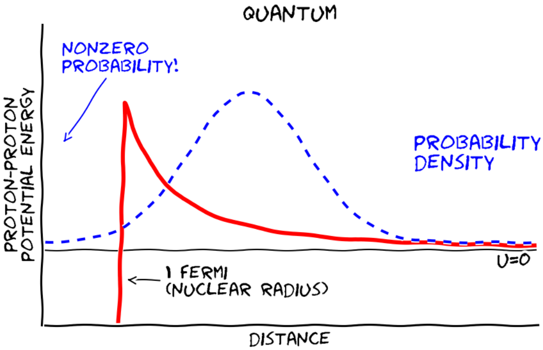
Why do pp chain reactions not take place in the ocean, where there is plenty of hydrogen?

- The **Coulomb barrier** prevents fusion. The average kinetic energy of oceanic H is *much* less than the height of the potential energy barrier due to electrostatic repulsion. Repulsion keeps the protons too far apart for strong interactions to take over.
- In fact, even at tens of millions of degrees, as in the centers of main sequence stars, the average kinetic energy is *still* too small for classical collisions to result in protons penetrating the Coulomb barrier.

Quantum mechanical **tunneling** through the Coulomb barrier is still required for fusion to occur, even at stellar core temperatures!

Fusion by tunneling





A classical particle of energy E encountering a Coulomb barrier of potential U will scatter off the barrier if $E < U$.

A quantum mechanical particle has a nonzero probability of penetrating the barrier because its wave function has a nonzero evanescent component in the region $E < U$.

Temperature dependence of proton fusion rate

Let us do a simplified version of a calculation first done by George Gamow (Gamow 1928) and also in Ryden pp. 362-366 and Shu p. 115.

Consider in 1D the fusion of two particles with masses m_1 and m_2, charges q_1 and q_2, speeds v_1 and v_2, and separation r. Their reduced mass and relative speed is

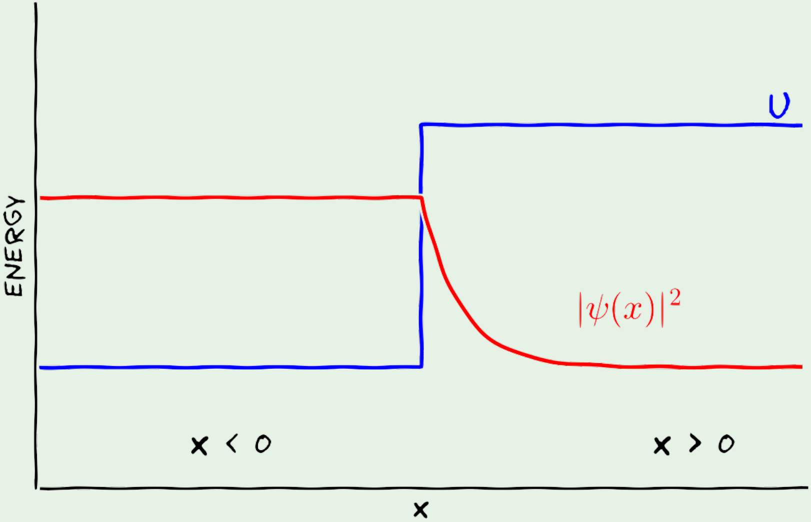
$$m = \frac{m_1 m_2}{m_1 + m_2}, \qquad v = v_1 - v_2$$

Classically, they cannot get closer together than $r_{\min}$, where

$$W = \frac{1}{2}mv^2 = \frac{q_1 q_2}{r_{\min}} \implies r_{\min} = \frac{2q_1 q_2}{mv^2}$$

Temperature dependence of proton fusion rate

Next, consider tunneling through a 1D infinite potential energy "step" barrier of height U.



Start with the Schrödinger Equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U\psi = E\psi, \quad E < U$$
$$\psi(x > 0) = \frac{2k_1}{k_1 + k_2} e^{ik_2 x}$$

where

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}} \qquad \text{and} \qquad k_2 = i\kappa_2 = \sqrt{\frac{2m(E-U)}{\hbar^2}}$$

The probability density that the particle is in the non-classical region $x > 0$ is nonzero:

$$p(x) = |\psi|^2 = \psi^* \psi = \frac{4k_1^2}{k_1^2 - \kappa_2^2} e^{-2\kappa_2 x}$$

The probability density p, integrated over the well on the other side of the barrier, gives the probability that an incident particle with energy E will be found there, on the other side. Note that

$$\begin{aligned} p &\propto \exp\left(-2x\sqrt{\frac{2mE}{\hbar^2}}\right) = \exp\left[-2x\sqrt{2m\frac{4\pi^2}{h^2}\frac{mv^2}{2}}\right] \\ &\propto \exp\left[-\frac{4\pi xmv}{h}\right] \\ &= \exp\left[-\frac{4\pi x}{\lambda}\right] \end{aligned}$$

where

$$\lambda = \frac{h}{mv}$$

is the de Broglie wavelength of the incident particle.

So the probability of tunneling past a barrier of width r_{min} can be written as

$$P_{\text{tunnel}} = B \exp\left[-\frac{4\pi r_{\text{min}}}{\lambda}\right] = B \exp\left[-\frac{8\pi q_1 q_2}{hv}\right]$$

where B is a constant. The probability that two particles have relative speed v is given by the Maxwell-Boltzmann distribution:

$$P(v) = C \exp\left[-\frac{mv^2}{2kT}\right]$$

where C is another constant. Do not worry about the value of these constants for now.

The probability of having tunneling and speed v is the product of these two probabilities:

$$P = D \exp\left[-\frac{8\pi q_1 q_2}{hv} - \frac{mv^2}{2kT}\right]$$

The fusion rate is proportional to this probability P. For what v is the rate largest? Find by setting the derivative equal to zero:

$$\frac{dP}{dv} = D\exp\left[-\frac{8\pi q_1 q_2}{hv} - \frac{mv^2}{2kT}\right]\frac{d}{dv}\left(-\frac{8\pi q_1 q_2}{hv} - \frac{mv^2}{2kT}\right)$$

$$0 = D\exp\left[-\frac{8\pi q_1 q_2}{hv} - \frac{mv^2}{2kT}\right]\left(\frac{8\pi q_1 q_2}{hv^2} - \frac{mv}{kT}\right)$$

This will be true when

$$\frac{8\pi q_1 q_2}{hv^2} = \frac{mv}{kT}$$

$$v = \left(\frac{8\pi q_1 q_2 kT}{hm}\right)^{1/3}$$

This value of v gives the fastest fusion rate.

Taking the parameters for pp fusion,

$$m = \frac{m_p}{2} = 0.84 \times 10^{-24}\ \mathrm{g}, \qquad q_1 = q_2 = 4.803 \times 10^{-10}\ \mathrm{esu}$$

and we get, for $T = 15.7$ MK,

$$v = \left(\frac{8\pi q_1 q_2 kT}{hm}\right)^{1/3} = 1.3 \times 10^{8}\ \mathrm{cm/s}$$

$$r_{\mathrm{min}} = \frac{2q_1q_2}{mv^2} = 2.4 \times 10^{-11}\ \mathrm{cm} \gg 10^{-13}\ \mathrm{cm}$$

Finally,

$$
\begin{aligned}
P_{\max} &= D \exp\left[-\frac{8\pi q_1 q_2}{hv} - \frac{mv^2}{2kT}\right] \\
&= D \exp\left[-\frac{8\pi q_1 q_2}{h}\left(\frac{8\pi q_1 q_2 kT}{hm}\right)^{-1/3} - \frac{m}{2kT}\left(\frac{8\pi q_1 q_2 kT}{hm}\right)^{2/3}\right] \\
&= D \exp\left[-\left(\frac{8\pi q_1 q_2 kT}{hm}\right)^{2/3}\left(\frac{m}{kT}\right) - \left(\frac{8\pi q_1 q_2 kT}{hm}\right)^{2/3}\left(\frac{m}{2kT}\right)\right] \\
&= D \exp\left[-\left(\frac{8\pi q_1 q_2 kT}{hm}\right)^{2/3}\left(\frac{3m}{2kT}\right)\right] \\
&= D \exp\left[-\left(\frac{T_0}{T}\right)^{1/3}\right]
\end{aligned}
$$

Temperature dependence of proton fusion rate

Note that we defined

$$T_0 = \left(\frac{3}{2}\right)^3 \left(\frac{8\pi q_1 q_2}{h}\right)^2 \frac{m}{k} = 1.565 \times 10^{10}\ \mathrm{K}$$

The fusion rate is quite sensitive to temperature. Some numbers:

Location	T	$P_{\mathbf{max}}/D$
Earth	300 K	5×10^{-163}
Sun	15.7 MK	4.6×10^{-5}
Massive star	100 MK	4.6×10^{-4}