Inside Living Stars, and Dead & Failed Stars

Nucleosynthesis Solar neutrinos Pulsating stars & the instability strip Helioseismology The standard solar model Electron and Neutron Degeneracy Pressure White Dwarfs and Neutron Stars Brown Dwarfs and Giant Planets

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Inside Living Stars, and Dead & Failed Stars

- Temperature dependence of the fusion rate in stars
- Nucleosynthesis and the cosmic abundances of the elements
- Solar neutrinos and the former solar neutrino problem
- Radial and nonradial pulsations in stars
- Pulsating stars and the instability strip
- Helioseismology
- The standard solar model
- Degeneracy pressure of electrons and neutrons
- White dwarfs
- Neutron stars
- Brown dwarfs and giant planets

Reading: Kutner Ch. 10.4 & 11.2–11.3, Ryden Sec. 18.1–18.2



HST image of the A1 star Sirius A (overexposed in center) and its white-dwarf companion Sirius B (lower left). From NASA/HST.

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Temperature dependence of proton fusion rate

Let us do a simplified version of a calculation first done by George Gamow (Gamow 1928) and also in Ryden pp. 362-366 and Shu p. 115.

Consider in 1D the fusion of two particles with masses m_1 and m_2 , charges q_1 and q_2 , speeds v_1 and v_2 , and separation r. Their reduced mass and relative speed is

$$m = rac{m_1 m_2}{m_1 + m_2}, \qquad v = v_1 - v_2$$

Classically, they cannot get closer together than r_{\min} , where

$$W = \frac{1}{2}mv^2 = \frac{q_1q_2}{r_{\min}} \implies r_{\min} = \frac{2q_1q_2}{mv^2}$$

Temperature dependence of proton fusion rate



Start with the Schrödinger Equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U\psi = E\psi, \quad E < U$$
$$\psi(x > 0) = \frac{2k_1}{k_1 + k_2}e^{ik_2x}$$

where

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}}$$
 and $k_2 = i\kappa_2 = \sqrt{\frac{2m(E-U)}{\hbar^2}}$

The probability density that the particle is in the non-classical region x > 0 is nonzero:

$$p(x) = |\psi|^2 = \psi^* \psi = \frac{4k_1^2}{k_1^2 - \kappa_2^2} e^{-2\kappa_2 x}$$

Set the width of the barrier to r_{\min} , after which the potential energy drops back to zero. The probability density p, integrated over the well on the other side of the barrier, then gives the probability that an incident particle with energy E will be found at $x > r_{\min}$:

$$p \propto \exp\left(-2r_{\min}\sqrt{\frac{2mE}{\hbar^2}}\right) = \exp\left[-2r_{\min}\sqrt{2m\frac{4\pi^2}{h^2}\frac{mv^2}{2}}\right]$$
$$\propto \exp\left[-\frac{4\pi r_{\min}mv}{h}\right] = \exp\left[-\frac{4\pi r_{\min}}{\lambda}\right]$$

where $\lambda = \frac{h}{mv}$ is the de Broglie wavelength of the incident particle. So the probability of tunneling past a barrier of width r_{\min} can be written as

$$P_{\text{tunnel}} = B \exp\left[-\frac{4\pi r_{\min}}{\lambda}\right] = B \exp\left[-\frac{8\pi q_1 q_2}{hv}\right]$$

where *B* is a constant.

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If the particles are in thermal equilibrium at temperature T, then the probability that two particles have relative speed v is given by the Maxwell-Boltzmann distribution:

$$P(v) = C \exp\left[-\frac{mv^2}{2kT}\right]$$

where *C* is another constant. Do not worry about the value of these constants for now.

The probability of having tunneling and speed v is then product of these two probabilities:

$$P = D \exp\left[-\frac{8\pi q_1 q_2}{hv} - \frac{mv^2}{2kT}\right]$$

The fusion rate is proportional to this probability *P*. For what *v* is the rate largest? Find by setting the derivative equal to zero:

$$\frac{dP}{dv} = D \exp\left[-\frac{8\pi q_1 q_2}{hv} - \frac{mv^2}{2kT}\right] \frac{d}{dv} \left(-\frac{8\pi q_1 q_2}{hv} - \frac{mv^2}{2kT}\right)$$
$$0 = D \exp\left[-\frac{8\pi q_1 q_2}{hv} - \frac{mv^2}{2kT}\right] \left(\frac{8\pi q_1 q_2}{hv^2} - \frac{mv}{kT}\right)$$

This will be true when

$$\frac{8\pi q_1 q_2}{hv^2} = \frac{mv}{kT}$$
$$v = \left(\frac{8\pi q_1 q_2 kT}{hm}\right)^{1/3}$$

This value of v gives the fastest fusion rate.

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Taking the parameters for pp fusion,

$$m = \frac{m_p}{2} = 0.84 \times 10^{-24} \text{ g}, \qquad q_1 = q_2 = 4.803 \times 10^{-10} \text{ esu}$$

and we get, for T = 15.7 MK,

$$v = \left(\frac{8\pi q_1 q_2 kT}{hm}\right)^{1/3} = 1.3 \times 10^8 \text{ cm/s}$$

 $v_{\min} = \frac{2q_1 q_2}{mv^2} = 2.4 \times 10^{-11} \text{ cm} \gg 10^{-13} \text{ cm}$

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The maximum probability, and therefore the maximum fusion rate, is then

$$\begin{aligned} & = D \exp\left[-\frac{8\pi q_1 q_2}{hv} - \frac{mv^2}{2kT}\right] \\ &= D \exp\left[-\frac{8\pi q_1 q_2}{h} \left(\frac{8\pi q_1 q_2 kT}{hm}\right)^{-1/3} - \frac{m}{2kT} \left(\frac{8\pi q_1 q_2 kT}{hm}\right)^{2/3}\right] \\ &= D \exp\left[-\left(\frac{8\pi q_1 q_2 kT}{hm}\right)^{2/3} \left(\frac{m}{kT}\right) - \left(\frac{8\pi q_1 q_2 kT}{hm}\right)^{2/3} \left(\frac{m}{2kT}\right)\right] \\ &= D \exp\left[-\left(\frac{8\pi q_1 q_2 kT}{hm}\right)^{2/3} \left(\frac{3m}{2kT}\right)\right] \\ &= D \exp\left[-\left(\frac{8\pi q_1 q_2 kT}{hm}\right)^{2/3} \left(\frac{3m}{2kT}\right)\right] \end{aligned}$$

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Temperature dependence of proton fusion rate

Note that we defined

$$T_0 = \left(\frac{3}{2}\right)^3 \left(\frac{8\pi q_1 q_2}{h}\right)^2 \frac{m}{k} = 1.565 \times 10^{10} \text{ K}$$

The fusion rate is quite sensitive to temperature. Some numbers:

			10 6				
Location	Т	$P_{\rm max}/D$	$\bigcirc 10^{12}$	/			
Earth	300 K	5×10^{-163}	L 18				
Sun	15.7 MK	$4.6 imes10^{-5}$	10 10				
Massive star	100 MK	$4.6 imes10^{-4}$	10 24	-	1	1	-
			10	05	10^{6}	10^{7}	10^{8}
					Т (К)		

Implications

- Fusion is still fairly slow in the centers of stars. (Good!)
- Fusion rates are *very* sensitive to temperature and are *much* higher at higher temperatures.
- Fusion rates are much lower for larger values of nuclear charge. This is why the CNO cycles require higher temperatures than the pp chains in order to be significant energy sources.
- Nucleosynthesis: Fusion in stellar cores produces heavier elements out of hydrogen, in amounts that should tend to decrease with increasing atomic number and nuclear weight.

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Hotter fusion and heavier elements

In principle, could stars live forever simply by gravitationally contracting and increasing their temperature to ignite the next heavier source of nuclear fuel whenever they run out of the lighter elements?

- No. The strong interaction range is smaller than the diameters of all but the smallest nuclei, but the range of the Coulomb interaction still covers the whole nucleus.
- If nuclei get large enough, the increase in electrostatic repulsion of the protons becomes greater than the increase in the binding energy from the strong interaction.
- Thus, there is a peak in the relationship between binding energy per baryon vs. atomic mass number.

• The peak turns out to lie at iron (Fe: A = 56, Z = 26).

Hotter fusion and heavier elements



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Neutrino production in the pp cycle



Solar neutrinos produced in the pp chain in the Standard Solar Model (from Wikimedia Commons).

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The solar neutrino "problem"

The rate of emission of solar neutrinos should match the pp fusion rate at the center of the Sun. We know what the temperature is at the center, so we can accurately predict the neutrino rate. However, the **observed rate** of v_e is lower than the **produced rate** by factors of 2 to 4, depending on the energy. Why?

Possibilities:

- ▶ We are wrong about the temperature. *Very unlikely, as you will show in recitation.*
- The center has cooled significantly, less than one photon-diffusion time ago (31,000 yr). Very unlikely, based on solar oscillation measurements (as you will show in the homework).

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Neutrino oscillations

It turns out that neutrinos have a small amount of mass (~ 1 eV, or about $10^{-9}m_p$), and this allows them to change "flavor" from ν_e to ν_μ and ν_τ while in transit.



Long-range oscillations of an initial v_e *to a* v_{μ} *or a* v_{τ} *. From Wikimedia Commons.*

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Acoustic waves in stars

Interior pressure of a star with uniform density ρ

Because the star is in hydrostatic equilibrium,

$$\frac{dP}{dr} = -\frac{GM(r)\rho}{r^2} = -\frac{G\rho}{r^2}\frac{4}{3}\pi\rho r^3$$
$$= -\frac{4\pi}{3}G\rho^2 r$$

Remembering that the surface pressure P(R) = 0, we can integrate to solve for P(r):

$$P(R) - P(r) = \int_{r}^{R} \frac{dP}{dr'} dr'$$
$$P(r) = \frac{4\pi}{3} G\rho^{2} \int_{r}^{R} r' dr'$$
$$= \frac{2\pi}{3} G\rho^{2} \left(R^{2} - r^{2}\right)$$

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Acoustic waves in stars

Pressure waves move at the speed of sound. The adiabatic speed of sound is

$$v_s = \sqrt{rac{\gamma P}{
ho}} = \sqrt{rac{\gamma kT}{\mu}}$$

where $\gamma \equiv C_P/C_V$ is the adiabatic index. For monatomic gases, like the ionized gases in stars, $\gamma = 5/3$.

If sounds waves can propagate in a medium of finite size, then it is possible for there to be **standing waves** of pressure due to the reflection of sound from the "ends" of the object.

Sharp changes in sound speed can result in reflection of sound, like a sharp change in P/ρ at the surface of a star, its center, the edge of its convection zone, etc.

Acoustic waves in a clarinet

Notes played by wind instruments correspond to acoustic standing waves. In a clarinet, the standing waves have a pressure node on the open end (bell) and an antinode on the closed end (mouthpiece).

