Inside Living Stars, and Dead & Failed Stars

Solar neutrinos
Pulsating stars & the instability strip
Helioseismology
The standard solar model
Electron and Neutron Degeneracy Pressure
White Dwarfs and Neutron Stars
Brown Dwarfs and Giant Planets

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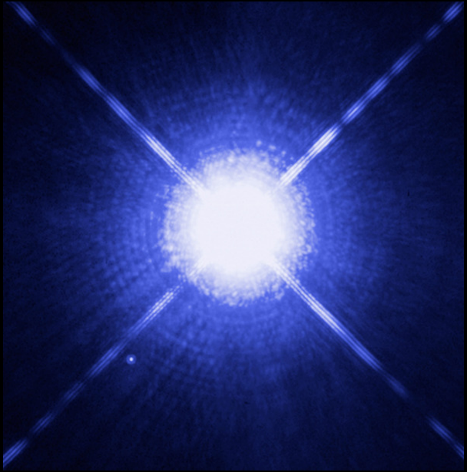
University of Rochester

Inside Living Stars, and Dead & Failed Stars

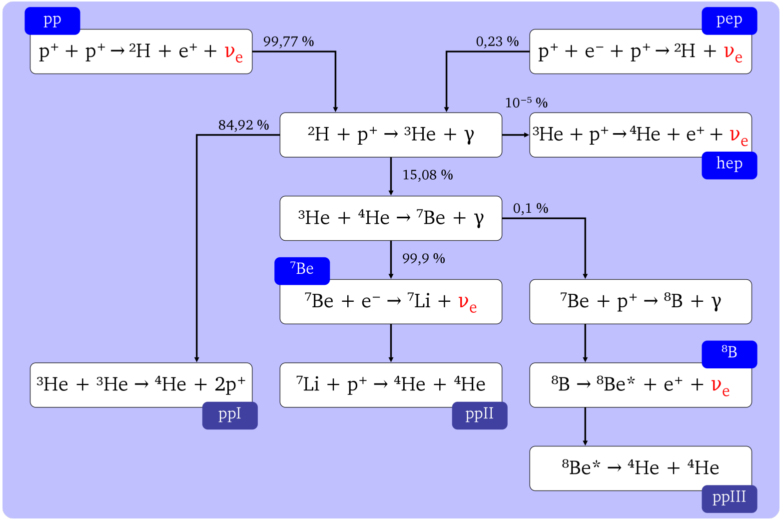
- Solar neutrinos and the former solar neutrino problem
- Radial and nonradial pulsations in stars
- Pulsating stars and the instability strip
- Helioseismology
- The standard solar model
- Degeneracy pressure of electrons and neutrons
- White dwarfs
- Neutron stars
- Brown dwarfs and giant planets

Reading: Kutner Ch. 10.4 & 11.2–11.3, Ryden Sec. 18.1–18.2



HST image of the A1 star Sirius A (overexposed in center) and its white-dwarf companion Sirius B (lower left). From NASA/HST.

Neutrino production in the pp cycle



Solar neutrinos produced in the pp chain in the Standard Solar Model (from Wikimedia Commons).

The solar neutrino "problem"

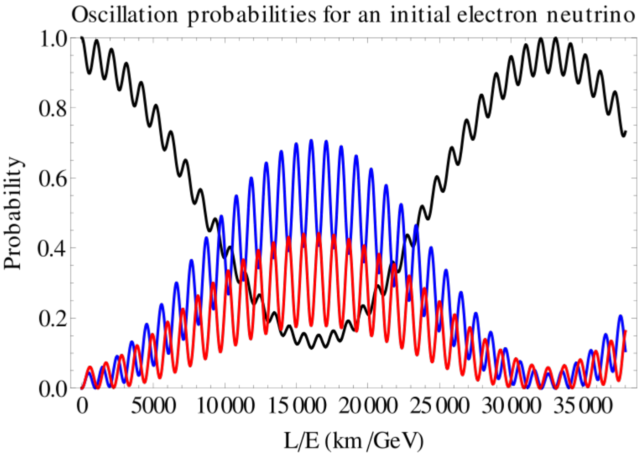
The rate of emission of solar neutrinos should match the pp fusion rate at the center of the Sun. We know what the temperature is at the center, so we can accurately predict the neutrino rate. However, the **observed rate** of ν_e is lower than the **produced rate** by factors of 2 to 4, depending on the energy. Why?

Possibilities:

- We are wrong about the temperature. *Very unlikely, as you showed in recitation.*
- The center has cooled significantly, less than one photon-diffusion time ago (31,000 yr). *Very unlikely, based on solar oscillation measurements (as you will show in the homework).*

Neutrino oscillations

It turns out that neutrinos have a small amount of mass (~ 1 eV, or about $m_p \times 10^{-9}$), and this allows them to change "flavor" from ν_e to ν_μ and ν_τ while in transit.



Long-range oscillations of an initial ν_e to a ν_μ or a ν_τ. From Wikimedia Commons.

Acoustic waves in stars

Interior pressure of a star with uniform density ρ

Because the star is in hydrostatic equilibrium,

$$\frac{dP}{dr} = -\frac{GM(r)\rho}{r^2} = -\frac{G\rho}{r^2}\frac{4}{3}\pi\rho r^3$$
$$= -\frac{4\pi}{3}G\rho^2 r$$

Remembering that the surface pressure $P(R) = 0$, we can integrate to solve for $P(r)$:

$$P(R) - P(r) = \int_r^R \frac{dP}{dr'}\,dr'$$
$$P(r) = \frac{4\pi}{3}G\rho^2 \int_r^R r'\,dr'$$
$$= \frac{2\pi}{3}G\rho^2\left(R^2 - r^2\right)$$

Acoustic waves in stars

Pressure waves move at the speed of sound. The adiabatic speed of sound is

$$v_s = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{\gamma k T}{\mu}}$$

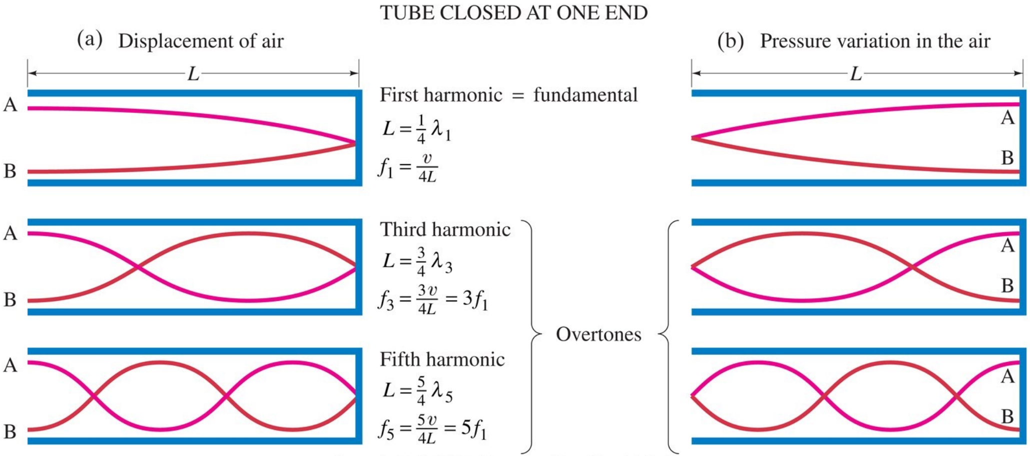
where $\gamma = C_P / C_V$ is the adiabatic index. For monatomic gases, like the ionized gases in stars, $\gamma = 5/3$.

If sounds waves can propagate in a medium of finite size, then it is possible for there to be **standing waves** of pressure due to the reflection of sound from the "ends" of the object.

Sharp changes in sound speed can result in reflection of sound, like a sharp change in P/ρ at the surface of a star, its center, the edge of its convection zone, etc.

Acoustic waves in a clarinet

Notes played by wind instruments correspond to acoustic standing waves. In a clarinet, the standing waves have a pressure node on the open end (bell) and an antinode on the closed end (mouthpiece).



Acoustic waves in a clarinet

What is the frequency and period of the fundamental mode of a clarinet?

$$f_1 = \nu_1 = \frac{v_s}{\lambda} = \frac{v_s}{4L}$$

Let $L = 60$ cm and $v_s = \sqrt{\gamma kT/\mu}$. For the air in this room,

- $T = 15°\mathrm{C} = 288$ K
- $\mu(\text{dry air}) = 0.78 m_{\mathrm{N_2}} + 0.21 m_{\mathrm{O_2}} + 0.01 m_{\mathrm{Ar}} = 4.81 \times 10^{-23}$ g
- $\gamma = 7/5$ (ideal diatomic gas at 288 K)
- Therefore, $v_s = 3.40 \times 10^4$ cm/s $= 761$ mph

Hence,

$$\nu_1 = \frac{34000 \text{ cm/s}}{4(60 \text{ cm})} \approx 142 \text{ Hz, or D below middle C}$$

$$\Pi_1 = \frac{1}{\nu_1} = \frac{4L}{v_s} = 7 \text{ ms}$$

Acoustic waves in stars

Now consider the "pipe" to be represented by the surface and center of a star. The "pipe" is *open* on the surface and *closed* at the center, since the compression wave cannot move to $r < 0$. Hence, the period of the longest standing wave is

$$\Pi = 4\int_0^R \frac{dr}{v_s} = \frac{4}{\sqrt{\frac{2\pi}{3}\gamma G\rho}}\int_0^R \frac{dr}{\sqrt{R^2 - r^2}} = 4\sqrt{\frac{3}{2\pi\gamma G\rho}}\int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

$$= 4\sqrt{\frac{3}{2\pi\gamma G\rho}}\int_0^{\pi/2} \frac{\cos u\, du}{\sqrt{1-\sin^2 u}}$$

$$= 4\sqrt{\frac{3}{2\pi\gamma G\rho}}\int_0^{\pi/2} du = 4\sqrt{\frac{3}{2\pi\gamma G\rho}}\frac{\pi}{2}$$

$$\boxed{\Pi = \sqrt{\frac{6\pi}{\gamma G\rho}}}$$

Acoustic waves in stars

For a uniform density star that is the mass and size of the Sun, we get

$$\Pi = \sqrt{\frac{6\pi}{\gamma G \rho}} = 11,700 \text{ s} = 183 \text{ min} = 3 \text{ hr}$$

and the period increases if the density of the star decreases.

Note that:

- Pressure waves manifest themselves as oscillations of surface temperature and radius of the star, which in turn cause oscillations of the magnitude of the star.
- The details of the periods and amplitudes of the oscillations are sensitive to the density, its variations, and the equation of state ($P - \rho$ relation) **inside** the star.
- There is good agreement with the "loudest" pulsating stars.

Loud sound waves in stars: Pulsators

Since the discovery of the pulsation of Mira (o Ceti) around 1600 and δ Cephei in 1784, eight types of pulsating stars have been found:

Type	**Period**	**Ampl.** (Δm)	**Notes**
Long-period (Mira) variables	100–700 days	2–7 mag	red giants
Classical (Pop I) Cepheids	1–50 days	0.5–1.5 mag	supergiants
W Virginis stars (Pop II Cepheids)	2–45 days	0.5–1.5 mag	giants
RR Lyrae stars	1–48 hr	1–1.5 mag	giants
δ Scuti stars	1–3 hr	0.1–0.6 mag	near the main sequence
β Cephei stars	1–48 hr	0.05–0.2 mag	near the main sequence
SX Phoenicis stars (Pop II δ Scus)	1–12 hr	0.01–0.15 mag	near the main sequence
ZZ Ceti stars	100–1000 s	0.1–0.3 mag	white dwarfs

More information: General Catalog of Variable Stars.

δ Cephei, a classical cepheid

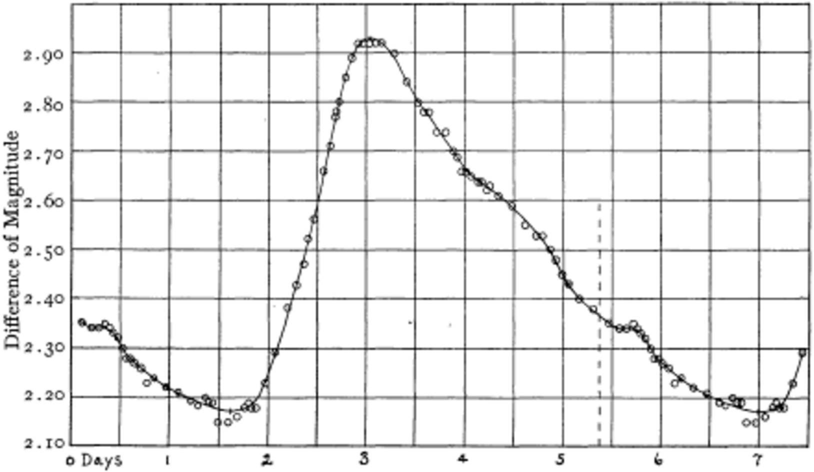
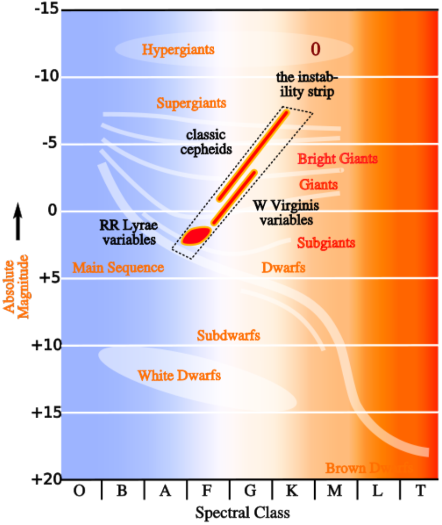


FIG. 1.—Light-Curve of δ Cephei.

The light curve of δ Cephei (Stebbins 1908), discovered to be variable in 1784.

Patterns of pulsators: The instability strip (IS)



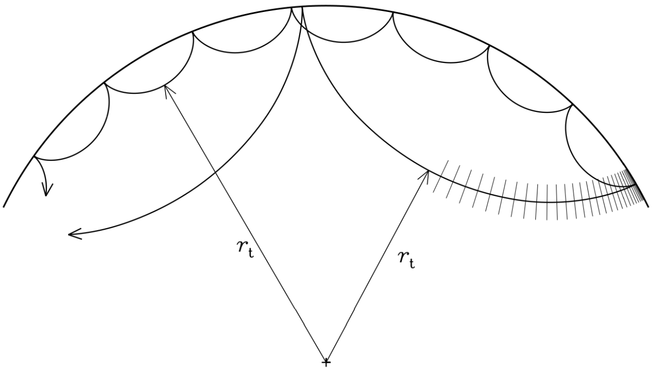
From Wikimedia commons.

Pulsating stars are not spread randomly in the luminosity-temperature space of the H-R diagram, nor are they all of a certain composition.

- Mira variables are supergiants on the **asymptotic giant branch**
- β Cep stars are all B stars lying just above the main sequence (MS).
- All the others lie along a long, narrow, nearly vertical patch of the H-R diagram at effective temperatures $\sim 10^4$ K; this is the instability strip (IS).
- δ Scu stars are close to the main sequence but most IS inhabitants are not there long; they develop rapidly after leaving the MS.

Nonradial stellar oscillations

- Some pressure waves may have transverse components as well as radial components.
- Such waves can be trapped between the surface and a layer of certain density in the interior as they propagate.
- At the surface, the waves reflect due to the density drop-off.
- In the interior, the waves refract, bending up due to the increase of v_s with depth.

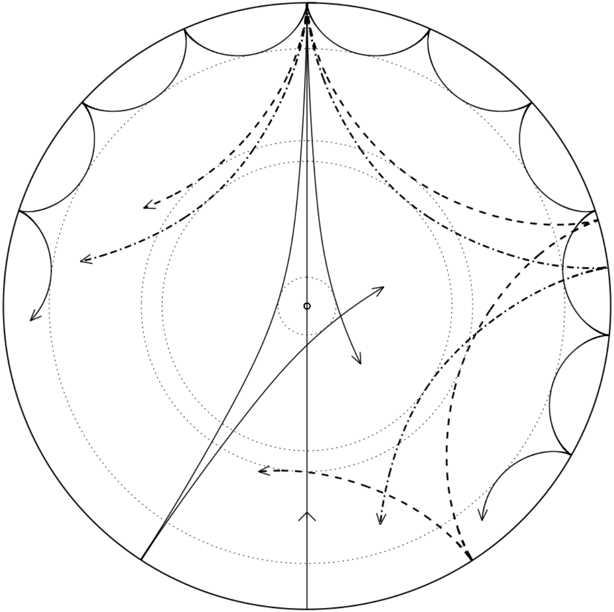


Propagation of acoustic waves corresponding to modes with $\ell = 30$ and $\nu = 3$ mHz (deeply penetrating rays) and $\ell = 100$, $\nu = 3$ mHz (shallowly penetrating rays) (Christiansen 2003).

Nonradial stellar oscillations

- The deeper the waves penetrate, the fewer surface reflections.
- This can still make standing waves, and thus stable oscillations, by having an integer number of wavelengths per circuit.

Modes with $\nu = 3$ *mHz in order of decreasing penetration depth:* $\ell = 0, 2, 20, 25,$ *and 75 (Christiansen 2003).*



Solar oscillations & helioseismology

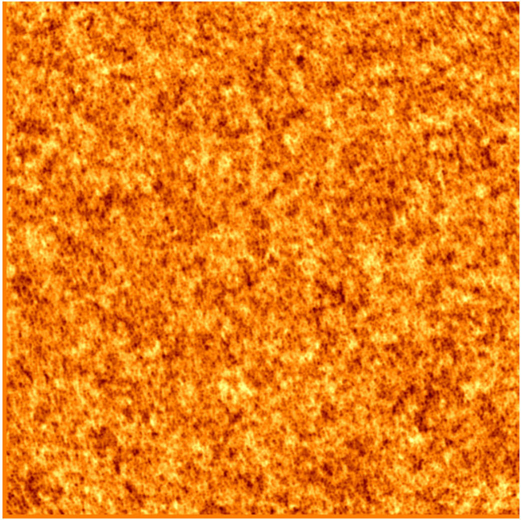
The Sun oscillates in many different modes with periods of $\sim$5 min. This was discovered in 1962 (Leighton et al. 1962) and later explained as nonradial oscillations (Ulrich 1970).

- It took a long time to notice these oscillations becuse they are so small. The velocity amplitudes are typically 100 cm/s with displacements of tens of meters.
- Many **thousands** of modes have now been identified.
- The period of each mode gives the integral of

$$\frac{1}{v_s} = \sqrt{\frac{\rho}{\gamma P}}$$

  along a different path through the Sun's interior.
- Different modes penetrate more deeply into the interior, providing good coverage for the outer 90% of the volume.

The solar oscillations



The Sun's nonradial oscillations observed on a 12-second cadence. From *SOHO/MDI (NASA)*

Precise knowledge of the solar interior

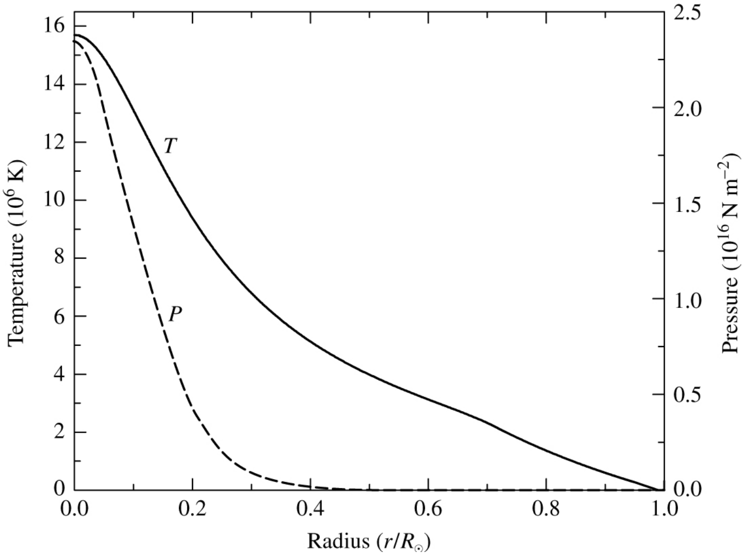
To summarize, we have:

- A direct measure of pp fusion per unit time at the very center of the Sun using measurements of solar neutrinos, corrected for neutrino oscillation.
- Thousands of measurements of integrals of

$$\frac{1}{v_s} = \sqrt{\frac{\rho}{\gamma P}}$$

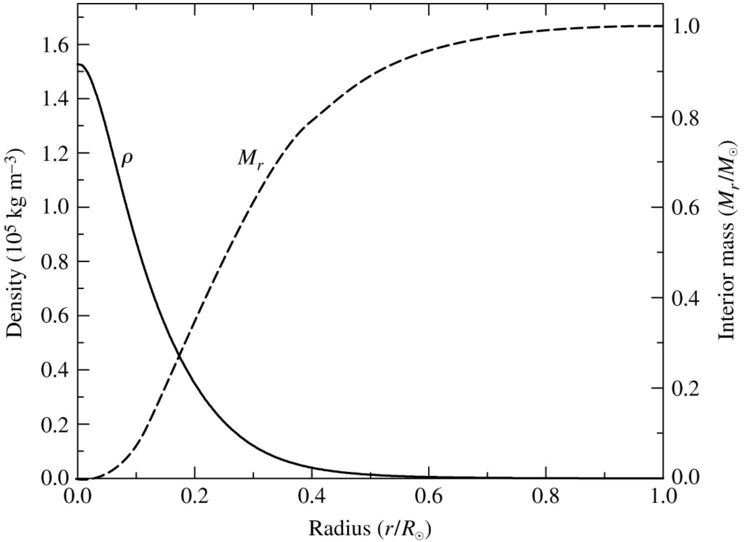
  from which the density, pressure, temperature, abundances, etc. can be determined over most of the solar volume.
- Confidence that we know precisely and accurately the thermodynamic parameters of all parts of the solar interior without being able to see the interior directly.
- The result is the Standard Solar Model (see, e.g., Bahcall et al. 2004).

The Standard Solar Model



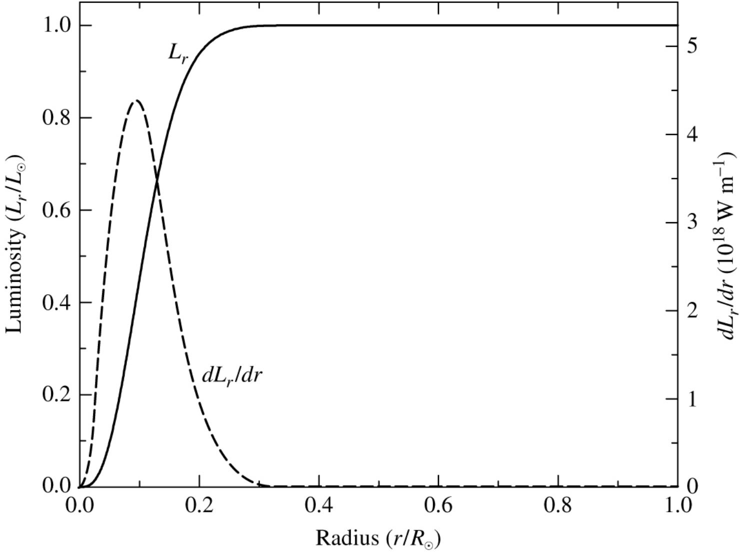
P(r) and T(r), from Carroll and Ostlie, An Introduction to Modern Astrophysics

The Standard Solar Model



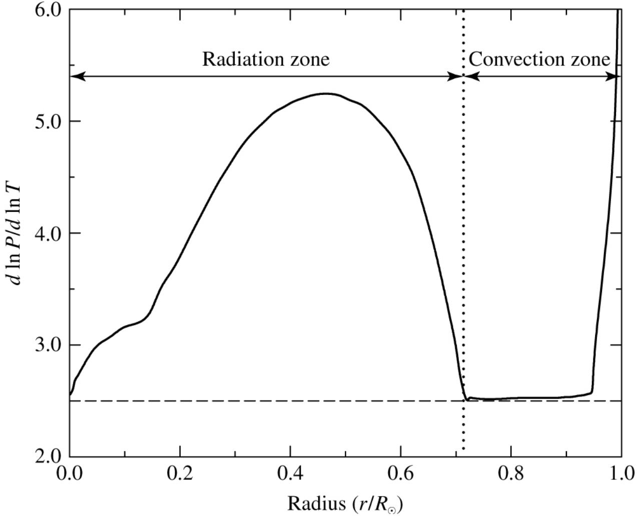
$\rho(r)$ and $M(r)$, from Carroll and Ostlie, An Introduction to Modern Astrophysics

The Standard Solar Model



Fusion power generation, from Carroll and Ostlie, An Introduction to Modern Astrophysics

The Standard Solar Model



Radiation and convection zones, showing the dominant energy transport mechanism for each region, from Carroll and Ostlie, An Introduction to Modern Astrophysics

How stars support themselves against gravity

Thermodynamics Gas and radiation pressure, for which the equations of state are

$$P_{\mathrm{gas}} = \frac{\rho k T}{\mu} = nkT \qquad\qquad P_{\mathrm{rad}} = \frac{4\sigma T^4}{3c}$$

support stars in which thermonuclear energy generation occurs.

Quantum Mechanics Degeneracy pressure sets in under extreme states of compression and/or low temperatures. As we will see, this implies quite different equations of state.

- Degeneracy pressure is the means of support for objects with no internal energy generation: dead stars like **white dwarfs** and **neutron stars**, unsuccessful stars (**brown dwarfs**), and the cores of giant planets (Jupiter, Saturn, etc.).

Degeneracy pressure

Degeneracy (or "exclusion") pressure (Fowler 1926) is due to:

1. The Pauli Exclusion Principle
   - No two identical **fermions** (spin-1/2 particles) can be in the same quantum state simultaneously.
2. The Heisenberg Uncertainty Principle
   - The uncertainties in the position and momentum of a particle are related by the inequality

$$\Delta x \Delta p_x \geq \frac{\hbar}{2}$$

   - An electron confined to a small region Δx has a relatively large uncertainty in its momentum Δp_x.

Degeneracy pressure

From these two principles we can make a hand-waving, non-rigorous derivation of the equation of state for degenerate matter. (See *Introduction to Quantum Mechanics* by Griffiths (Sec. 5.3.1) for a better derivation.)

Consider identical fermions of number density n. Each is inside its own little "box" of space with volume ℓ^3, such that

$$\ell = (1/n)^{1/3} \qquad \text{length of box}$$

$$\Delta x = \ell \qquad \text{each particle is somewhere inside its box}$$

If the particles have finite momentum, they hit the "walls" of their "boxes," which we will say have area A. They then exert a pressure

$$P \approx \frac{F}{A} = \frac{1}{A}\frac{dp}{dt} = \frac{1}{A} \underbrace{\frac{nA\delta x}{2}}_{\substack{\text{\# of pcls}\\ \text{hitting}\\ \text{wall}}} \underbrace{\frac{1}{\delta t}}_{\substack{\text{interval}\\ \text{of hit}}} \underbrace{p_x}_{\substack{\text{typical}\\ \text{momentum}\\ \text{per pcl.}}} = \frac{1}{2} n v_x p_x$$

Degeneracy pressure

Typical momentum per particle:

$$p_x \approx \Delta p_x \qquad \text{not exactly, but within a factor } \sim 1$$

$$\approx \frac{h}{\Delta x} \qquad \text{by the Uncertainty Principle}$$

$$= h n^{1/3} \qquad \text{cube bounding each particle}$$

Since the particles are moving in nonrelativistic motion, $p_x = m v_x$ and so

$$P = \frac{1}{2} n v_x p_x = \frac{n p_x^2}{2m} = \frac{h^2 n^{5/3}}{2m}$$

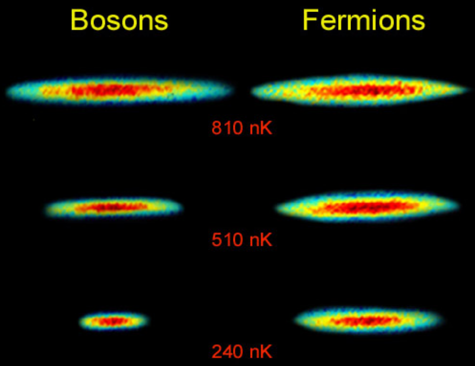
Done correctly using the Fermi-Dirac distribution for p, v, etc. gives the correct leading factor:

$$\boxed{P = 0.0485 \frac{h^2 n^{5/3}}{m}}$$

Degeneracy pressure in the lab

The cause of the Pauli Exclusion Principle and its physical limits are not understood, but it is well-demonstrated experimentally.

- Left cloud: ^{6}Li spin-1 **bosons**
- Right cloud: ^{7}Li spin-$\frac{3}{2}$ **fermions**
- As the temperature drops, the bosons bunch together…
- …but the cloud of fermions does not condense below 300 nK
- This is degeneracy pressure at work, driven by the Pauli Exclusion Principle



From Truscott & Hulet (2010)