Astronomy 142 — Recitation #8

Prof. Douglass

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Formulas to remember

Free-fall time

$$t_{ff} = \sqrt{\frac{3\pi}{32G\rho_0}} \tag{1}$$

"Pressure" from random motions of stars

$$P = \rho v_r^2 \tag{2}$$

where ρ is the mass density of stars (mass of stars per unit Galactic volume including the space between the stars, not the mass density of the stars themselves) and v_r is the random component of motion of a typical star, separate from any systematic (e.g. orbital) motion.

1D hydrostatic equilibrium

$$\frac{dP}{dz} = -\rho g_z \tag{3}$$

Mass per unit area in a hydrostatically-supported galactic disk

$$\mu = \frac{g_z}{2\pi G} = \frac{v_z^2}{2\pi G H} \tag{4}$$

Relaxation time

$$t_c = \frac{v^3}{4\pi G^2 m^2 n} \frac{1}{\ln\left(\frac{2R}{r}\right)} = \left(\frac{2R}{v}\right) \frac{N}{24 \ln\left(\frac{N}{2}\right)} \tag{5}$$

Virial theorem

In thermal equilibrium or steady state,

$$\frac{d^2 I}{dt^2} = 2K + U = K + E \tag{6}$$

where I, K, U, and E are the moment of inertia, total kinetic energy, total potential energy, and total mechanical energy of a closed system of particles whose interactions can be characterized by a scalar potential. It is often the case that $\frac{d^2 I}{dt^2} = 0$, for which the virial theorem simplifies to $K = -\frac{U}{2}$.

Rotation curves

$$v(r) = \sqrt{\frac{GM}{r}} \qquad \text{Point mass: Keplerian rotation}$$

$$v(r) = r\sqrt{\frac{4\pi G\rho_0}{3}} \qquad \text{Constant density: solid-body rotation}$$

$$v(r) = \sqrt{4\pi G\rho_0 r_0^2} = \text{constant} \qquad \frac{1}{r^2}\ \text{density distribution: flat rotation curve}$$

Abbreviations

- kpc $= 10^3$ pc
- Myr $= 10^6$ yr
- Gyr $= 10^9$ yr
- 1 km/s = 1.022 pc/Myr

Workshop problems

Remember! The workshop problems you will do in groups in Recitation are a crucial part of the process of building up your command of the concepts important in ASTR 142 and subsequent courses. Do not, therefore, do your work on scratch paper and discard it. Better for each of you to keep your own account of each problem in some sort of bound notebook.

1. A constant-density, spherical molecular cloud undergoes free-fall collapse. Consider molecules at two particular initial radii: those at the initial cloud radius r_0 and those halfway between the center and the edge, at $r_0/2$. Which molecules reach the center first? Explain (in detail).

2. In the Sun's neighborhood of the Galactic disk, suppose that the density can be thought of as an infinite planar distribution with a plane surface density μ (mass per unit disk area) and volume density ρ (mass per unit volume) that decreases sharply with increasing elevation z above or below $z = 0$.
   (a) Under these assumptions, explain why the equation of hydrostatic equilibrium can be written to good approximation as
   $$\frac{dP}{dz} = -\pi G\rho\mu \tag{7}$$
   (b) Solve this equation for the density as a function of z, and show that the density scale height obtained is the same as what we found in class.

3. *The number of pairs among N objects.* This result appears in so many contexts in physics that it is worth it to work through the derivation.
   (a) In how many distinctly different ways can N distinguishable objects be arranged? (If you prefer a concrete situation: Suppose that you have N books to put on a shelf. In how many different ways can you place them on the shelf?)
   (b) Now suppose that the N objects are indistinguishable from one another. In how many distinctly different ways can one arrange these objects? (For example, N identical books on a shelf.)
   (c) In how many distinctly different ways can one arrange N objects, of which n and $N - n$ are of two distinguishable types, but of which the n objects of the first type are indistinguishable from one another, and similarly the $N - n$ are indistinguishable from one another? (For example, n identical blue books and $N - n$ identical green books on a shelf.)
   (d) So how many different pairs are there among the N identical objects?

4. What is the total gravitational potential energy of a cluster of N stars that have typical mass m and typical separation r?

5. Use the virial theorem to show that a cluster of N stars with typical random velocity v and typical separation r has total mass $M = \frac{2rv^2}{G}$.

Intro to Python, lesson 8. (A feature *exclusive* of ASTR 142 recitations.)

6. Review your solution for computing the effect of the elliptical shape of the Earth's orbit on local sidereal time under the assumption that the orbit, though elliptical, lies in the Earth's equatorial plane.

   The real orbit (path of the Sun) does not lie in the equatorial plane at all; the Sun travels through the sky along the ecliptic instead of the celestial equator. These paths are not the same angular length; yet both the real and mean Sun get from equinox to solstice (6 hours of sidereal time) in a quarter of a year. Thus, the angular speed ω_0 of the real Sun varies along its track because of the tilt of the ecliptic.

   (a) For simplicity, suppose that the Earth's orbit is circular, but that the ecliptic and the celestial equator are tipped by $\psi = 23.5°$ and that they intersect at the vernal and autumnal equinoxes. Calculate the difference between the angular speeds of mean and real Sun, $\Delta\omega_0 = \omega - \omega_0$, at the vernal equinox.

   (b) Argue that $\Delta\omega_0$ has the same value at the vernal and autumnal equinoxes and the opposite of these values at the solstices, and that these are the maximum tilt-related differences between angular speed of the real and mean Sun.

   (c) Propose a simple form for $\Delta\omega_0(t)$ that satisfies the results from part b, and integrate it to obtain the angular difference $\Delta\theta_0(t)$ between the real and mean Sun.

   (d) Plot $\Delta\theta_0(t)$ over the course of a year, and compare the result to that obtained for the angular difference $\Delta\omega_\varepsilon(t)$ due to orbital eccentricity. Also plot the *net* angular difference between the real and mean Sun by adding these two results. This combined result is called the *equation of time* and relates sidereal time (or, if you like, sundial time) to clock time (mean solar time).