# Failed Stars & Stellar Remnants

Degeneracy pressure
White dwarfs & Neutron stars
Brown dwarfs & Giant planets
General relativity
Hawking radiation
Gravitational waves

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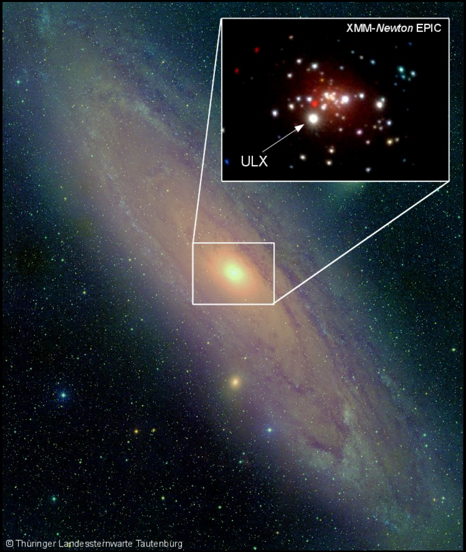
# Stellar-mass sized black holes

- Degeneracy pressure of electrons and neutrons
- White dwarfs
- Neutron stars
- Brown dwarfs and giant planets
- General relativity & its prediction of black holes
- Hyperspace
- Hawking radiation
- Gravitational radiation

**Reading**: Kutner Sec. 8.4 & 11.5, Ryden Sec. 18.3

*Right: HST image of the Andromeda galaxy with an X-ray image inset of an ultra-luminous X-ray source (a stellar-mass black hole). From the MPE.*



# Electron degeneracy pressure

Consider a gas of electrons, produced by ionization from atoms with nuclear charge $Ze$ and baryon number $A$. The electron and nuclear densities are related by

$$n_e = Zn_+$$

and the mass density is

$$\begin{aligned} \rho &= m_e n_e + A m_p n_+ && (m_p \approx m_n) \\ &\approx A m_p n_+ && (m_p \gg m_e) \end{aligned}$$

Therefore, $n_e = Z\rho/Am_p$ and the **equation of state** for electron degeneracy pressure is

$$\boxed{P_e = 0.0485 \frac{h^2}{m_e} \left(\frac{Z}{A}\right)^{5/3} \left(\frac{\rho}{m_p}\right)^{5/3}}$$

# Polytropic equation of state

Note that the equation of state is polytropic:

$$P \propto \rho^{\gamma}$$

where $\gamma = (n+1)/n$ is called the polytropic exponent and $n$ is called the polytropic index. This is one of the few functional forms for which the equations of stellar structure yield simple solutions. Example models:

| **Index** $n$ | **Notes** |
|---|---|
| $0.5 - 1$ | Neutron stars |
| 1.5 | Fully convective cores, brown dwarfs, gaseous planets |
| 3 | MS stars (Sun) and relativistic degenerate cores (WDs) |

As $n$ increases, the density distribution is more heavily weighted toward $r = 0$.

# Electron degeneracy in stars

Suppose we use the new EOS (equation of state) instead of the ideal gas law to balance gravity and support a star. Our former scaling relationships derived from the ideal gas law and gravity were

$$P_c \propto \frac{GM^2}{R^4} \qquad\qquad \rho_c \propto \frac{M}{R^3}$$

Precise calculations for the polytropic EOS $P \propto \rho^{5/3}$ and gravity turn out to give

$$P_c = 0.77\frac{GM^2}{R^4} \qquad\qquad \rho_c = 1.43\frac{M}{R^3}$$

We can now use this to estimate the typical size of a star balanced by electron degeneracy pressure.

# Size of an electron degenerate star

$$
\begin{aligned}
P_c &= P_e \\
0.77\frac{GM^2}{R^4} &= 0.0485\frac{h^2}{m_e}\left(\frac{Z}{A}\right)^{5/3}\frac{\rho^{5/3}}{m_p^{5/3}} \\
&= 0.0485\frac{h^2}{m_e}\left(\frac{Z}{A}\right)^{5/3}\left(\frac{1.43M}{m_p}\right)^{5/3}\frac{1}{R^5} \\
\therefore R &= 0.114\frac{h^2}{Gm_e m_p^{5/3}}\left(\frac{Z}{A}\right)^{5/3} M^{-1/3}
\end{aligned}
$$

A couple of points:

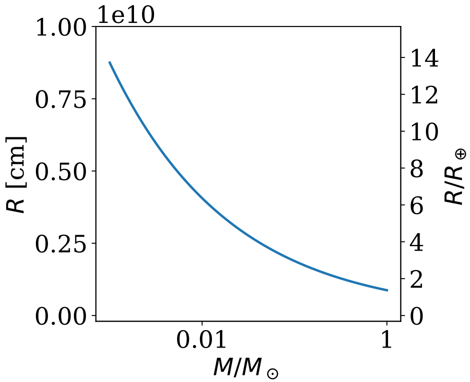
- There is no temperature dependence here! This is much more simple than a normal star.
- $R$ is much smaller than a normal star of equal mass.

# White dwarfs

## Numerical example

For $M = 1M_\odot$ and $Z/A = 0.5$, $R = 9 \times 10^8$ cm $\approx 1.4R_\oplus$. This object, a **white dwarf**, has the mass of a star and the size of a planet!

- Remarkable feature of $R$–$M$ relation: $R$ **decreases with increasing** $M$ (Stoner 1930).
- Cause: larger mass $M$ requires larger supporting $P_e$
- Larger $P_e$ implies larger electron momenta $p$ confined to a smaller "box" $\Delta x$. The entire object is smaller!



# Earth compared to the white dwarf star Sirius B

From NASA+ESA.

# Special Relativity & the white dwarf mass limit

- To support higher mass (smaller) white dwarfs, larger electron momenta (and speeds) are required.
- Special Relativity: electron velocities cannot exceed $c \approx 3 \times 10^{10}$ cm/s.
- Therefore, there is an upper bound (a maximum) to the mass that can be achieved by a white dwarf (Anderson 1929, Stoner 1930, Chandrasekhar 1931).
- Note: when $v \sim c$, $p$ is not simply $mv$. In this extreme relativistic limit,

$$\boxed{P_e = 0.123 hc n_e^{4/3}}$$

- Note that the relativistic and nonrelativistic expressions for $P_e$ are equal at $n_e = 10^{30}$ cm$^{-3}$, about the density of the core of a $0.3 M_\odot$ white dwarf (Stoner 1930).

# Stoner-Anderson-Chandrasekhar mass ("Chandrasekhar limit")

For $P \propto \rho^{4/3}$ and gravity, the central pressure and density of a star are

$$P_c = 11\frac{GM^2}{R^4} \qquad\qquad \rho_c = 12.9\frac{M}{R^3}$$

Balance $P_c$ with the relativistic electron degeneracy pressure $P_e$ and the radius disappears from the equation (you will show this in your homework). The result is a maximum mass known as the **Stoner-Anderson-Chandrasekhar mass** or the **Chandrasekhar limit**:

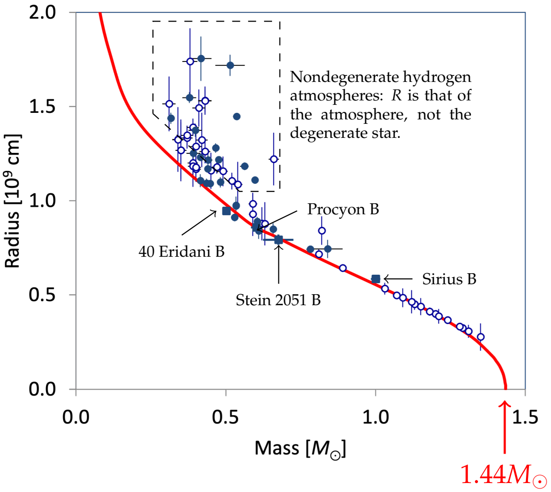
$$M_{\rm SAC} = 0.2\left(\frac{Z}{A}\right)^2\left(\frac{hc}{Gm_p^2}\right)^{3/2} m_p = \boxed{1.44 M_\odot}$$

for $Z/A = 0.5$.

# Theory vs. observation for white dwarfs

White dwarfs are hard to detect in binaries because they are so much fainter than main-sequence stars.

- On the right are white dwarfs in binaries with precisely known orbits (Provencal et al. 1998, 2002, Rebassa-Mansergas et al. 2012, Parsons et al. 2013, Sahu et al. 2017, Parsons et al. 2017).
- The white dwarfs in visual binaries are labeled with their names.



# Neutron stars

What happens in a dead star with $M > M_{\rm SAC}$?

- Such a star simply cannot be supported by electron degeneracy pressure. Add a little too much mass and it will collapse gravitationally or explode.
- During the collapse, the extra energy liberated from gravity, plus the high density, can help drive some endothermic nuclear reactions, notably

$$\Delta E + e^- + p \rightarrow n + \nu_e$$

- But **neutrons are also fermions**, and neutron degeneracy pressure can also balance gravity. A neutron star is formed (Tolman 1939, Oppenheimer & Volkov 1939).
- The nonrelativistic $R$-$M$ relation for a NS is

$$\boxed{R = 0.0685 \frac{h^2}{G m_p^{8/3}} M^{-1/3}}$$

This is the mass of a star and the size of a **city**.

# Size of a $1M_{\odot}$ neutron star

# Pulsars

- Most of the neutron stars we observe appear to us as pulars.
- A pulsar is a rapidly rotating NS (or WD) that emits a beam of electromagnetic radiation.
- Neutron stars can also be observed as X-ray flaring members of *low-mass binary systems* (LMXBs).
- PS B1919+21 was the first pulsar to be discovered in 1967 (Hewish et al. 1968)
- Right: the center of the Crab Nebula (M1), the remnant of SN 1054.

*Optical/X-ray overlay from the Hubble and Chandra telescopes.*

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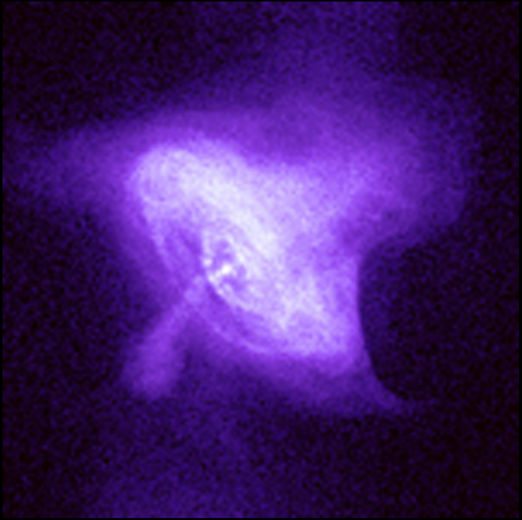
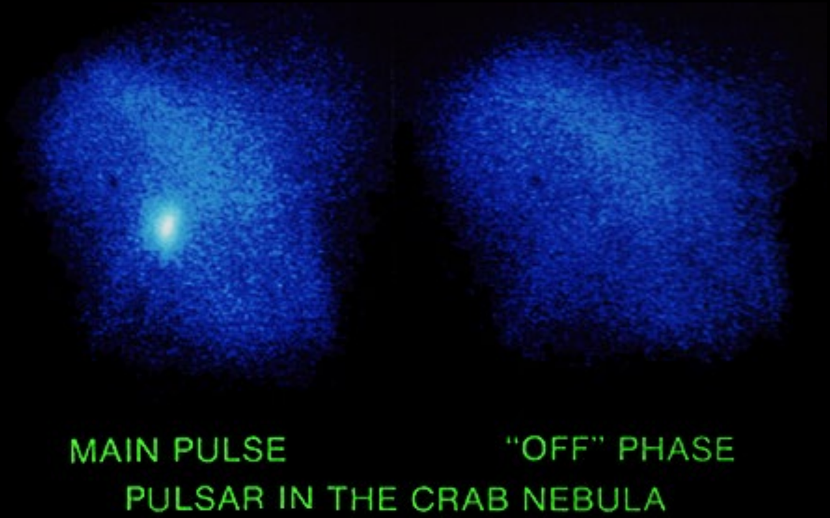


*Optical/X-ray overlay from the Hubble and Chandra telescopes.*

# Pulsars



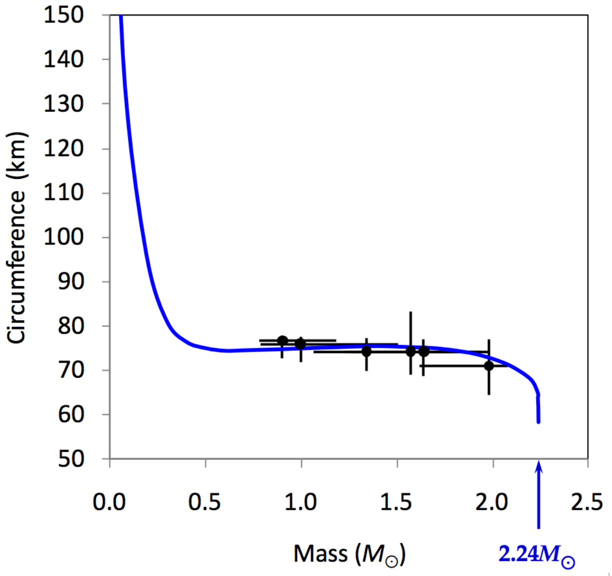

*X-ray images of the Crab Nebula from the Chandra X-ray Observatory (CXO) taken 0.015 s apart.*

# Neutron star masses

- The maximum mass of a neutron star is not easy to calculate as it requires General Relativity and a strong-interaction equation of state.
- The maximum mass turns out to be about $2.2 M_\odot$ (Lattimer 2012).
- The maximum cannot be $> 3M_\odot$ or the equation of state would imply a sound speed $v_s > c$.
- A handful of neutron stars are observed in low-mass X-ray binary (LMXB) systems and their masses and radii have been measured (see e.g. Steiner et al. 2010).
- Neutron stars are so small that eclipses enabling the determination of $R$ from timing are extremely rare. Instead, we get $R$ from the luminosity and temperature determined from spectra taken in between flares ("quiescent LMXB").

# Neutron star *R*–*M* relation

- The theoretical calculation of the NS *R*–*M* relation is beyond the scope of an undergraduate course.
- Currently the calculation of Lattimer & Prakash (2007) comes close to matching the data on neutron star binaries (Steiner et al. 2010).
- Note that circumference is plotted instead of radius.

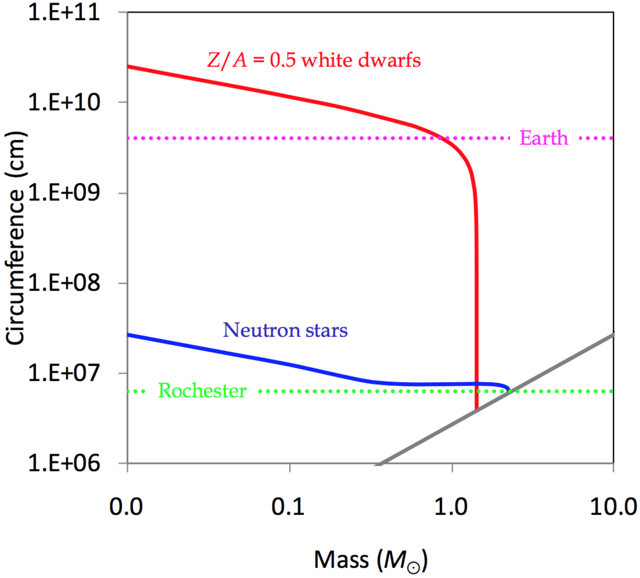


# Degenerate star $R$–$M$ relations

**Gray line**: causality cutoff
A star smaller than $5.8\pi GM/c^2$ turns out to require an EOS with

$$\frac{\partial P}{\partial \rho} > c^2$$

which in turn implies a speed of sound $v_s > c$, violating the order of cause and effect.

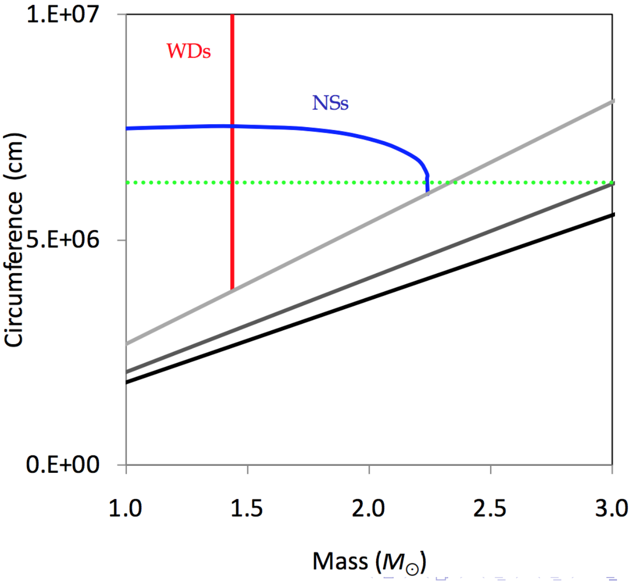


# Degenerate star $R$–$M$ relations

Below these lines, the interior of a stable star would have

- **Gray line**: $v_s > c$
- **Dark gray line**: $P > \infty$
- **Black line**: event horizon (would be a black hole)

In all these systems, the very large masses and small distances require a General Relativistic treatment.



# Far below $M_{\mathrm{SAC}}$: Brown dwarfs and giant planets

When stars form, they **contract** until they are hot enough in the center (about $3 \times 10^6$ K) to ignite the pp chain fusion reactions. Recall that

$$T_c = \frac{P_c \mu_c}{\rho_c k} \approx \frac{\mu_c}{k} \left( \frac{GM^2}{R^4} \right) \left( \frac{1}{150} \frac{R^3}{M} \right)$$
$$= 15.7 \times 10^6 \left( \frac{M}{M_\odot} \right) \left( \frac{R_\odot}{R} \right) \text{ K}$$

for solar-type stars, if gravity is supported by gas pressure.

- For **small** masses this involves gas pressures that become smaller than the electron degeneracy pressure, so degeneracy pressure can stop the contraction and prevent the object from reaching fusion temperatures. This imposes a lower mass limit on what can become a star.
- The **H-burning limit** turns out to be $0.08 M_\odot$.

# Far below $M_{SAC}$: Brown dwarfs and giant planets

Depending upon how they are formed and what their mass is, such objects are called either **brown dwarfs** or **giant planets**.

- Because they cannot replace the energy that leaks away in the form of radiation, they simply remain at the size determined by degeneracy pressure and cool off forever.
- Thus, if they are very old, they are *very* faint. This prevented their detection until 1995.
- Today, thousands are known from deep near-IR surveys and from Spitzer Space Telescope observations.
- Once it was thought that these objects could be numerous enough to comprise a significant (and invisible) component of the mass in the Galaxy.
- We will come back to this idea when we discuss **Dark Matter**.

# Escape velocities from stars

Neglecting relativity, an increasingly bad approximation:

$$\begin{aligned} E &= \frac{1}{2}mv_{\rm esc}^2 - \frac{GMm}{R} = 0 \\ \therefore v_{\rm esc} &= \sqrt{\frac{2GM}{R}} \\ &= 619\ \text{km/s} = 0.002c && \text{Sun} \\ &= 6970\ \text{km/s} = 0.023c && 1M_\odot\ \text{white dwarf} \\ &= 154000\ \text{km/s} = 0.514c && 1M_\odot\ \text{neutron star} \end{aligned}$$

And note that $v_{\rm esc} = c \approx 3 \times 10^5$ km/s when

$$R = \frac{2GM}{c^2}$$

# Beyond the NS maximum mass: Black holes

- The maximum mass of a neutron star is $\sim 2.2 M_\odot$. There is **no known physical process** that can support a heavier object without internal energy generation.
- A non-spinning heavier object will collapse past neutron-star dimensions and soon thereafter become a black hole, an object from which even light cannot escape if emitted within a distance

$$R_{\text{Sch}} = \frac{2GM}{c^2} \qquad \text{Schwarzschild radius}$$

  of the object *as measured by a distant observer*.
- This spherical surface is called the **event horizon** or simply the "horizon" of the black hole.
- The nonrelativistic result $R = 2GM/c^2$ is, by accident, the same as $R_{\text{Sch}}$ derived with the general theory of relativity (GR).

# Black holes & General Relativity

First published by Albert Einstein in 1916, GR is a description of the effect of gravity at any strength, even handling large amounts of mass shrunk to small dimensions. It involves new mathematical concepts beyond the scope of this course.

The theory provides a set of **field equations** to describe gravity:

$$\text{spacetime curvature} \rightarrow \boxed{G_{\mu\nu}} = \frac{8\pi G}{c^2} \boxed{T_{\mu\nu}} \leftarrow \text{mass and energy}$$

In plain English (*Gravitation* by Misner, 1973):

- Mass causes space and time to be **curved**, or **warped**.
- The resulting curvature of space determines **how masses will move**.

# Gravitational time dilation

Time and space are substantially warped near black holes.

- To distant observers, time intervals on clocks near black holes appear to be slow compared to their own local identical clocks.
- This effect is known as **gravitational time dilation** or **gravitational redshift**:

$$\text{time measured by a distant observer} \rightarrow \boxed{\Delta t} = \frac{\Delta\tau}{\sqrt{1-\frac{R_{\rm Sch}}{r}}} > \boxed{\Delta\tau} \leftarrow \text{proper time near the BH}$$

- Thus, *to a distant observer*, **time appears to stop at the event horizon**:

$$\Delta t \rightarrow \infty \text{ as } r \rightarrow R_{\rm Sch}$$

This behavior gave the horizon its original name: the "Schwarzschild singularity."

# Gravitational length dilation

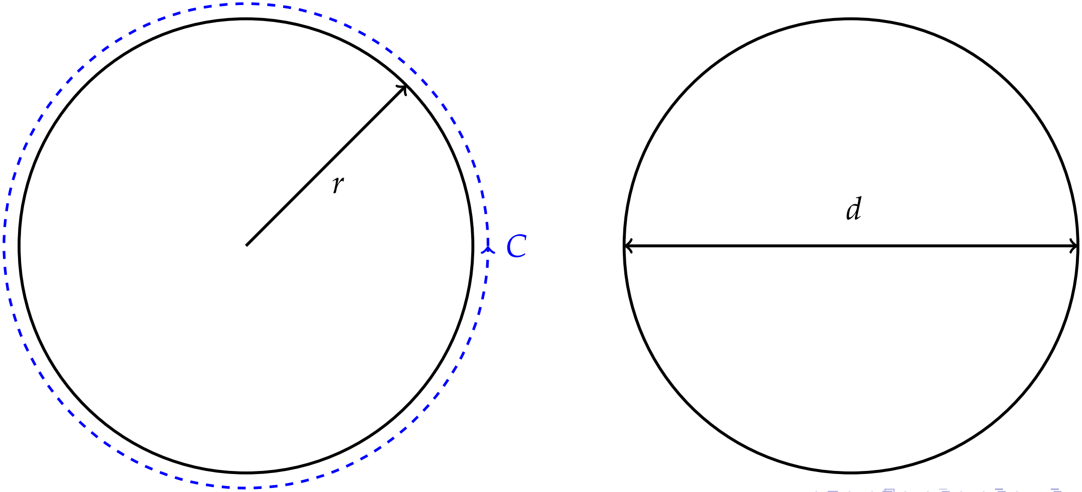
- Near a black hole, a *small* length $\Delta\mathcal{L}$ measured simultaneously (as with a measuring tape) between two points on a radial line is greater than the distance $\Delta r$ between the points, measured by a distant observer:

$$\text{proper length near the BH} \rightarrow \Delta\mathcal{L} = \frac{\Delta r}{\sqrt{1 - \frac{R_{\text{Sch}}}{r}}} > \Delta r \leftarrow \text{length measured by a distant observer}$$

- Neither $\Delta r$ nor the radius $r$ measured by a distant observer of a point near the black hole has meaning as a physical distance to an observer near the black hole.
  - $r$ and $\Delta r$ are called **coordinate distances**.
- However, the **circumference** $C$ of a circle through that point and centered on the black hole **turns out to have the same value in all frames**. Think of $r$ only as $r = \frac{C}{2\pi}$.

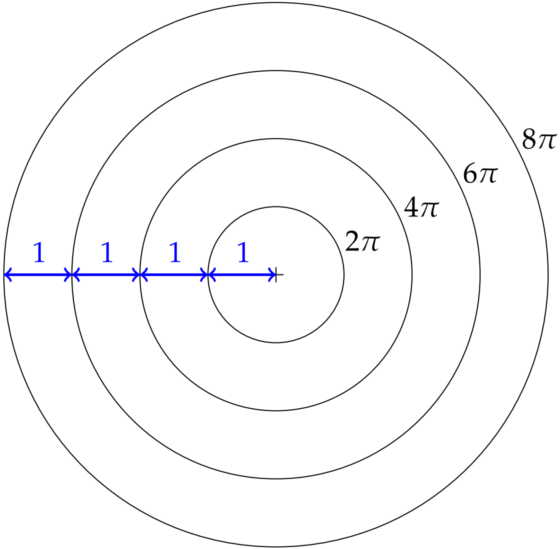
# Circular orbits and their radii in GR

Circles in flat spacetime: $C = 2\pi r = \pi d$. That, of course, is the very definition of $\pi$.



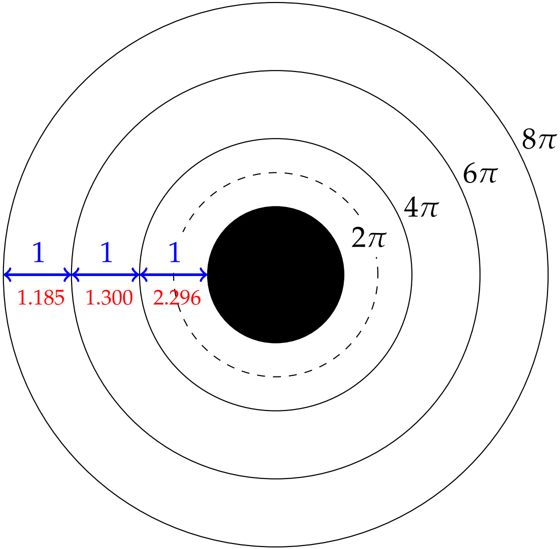
# Circular orbits in flat space (all in same plane)

- Concentric circular orbits in flat space.
- The distance between each orbit is 1.
- Local and distant observers report the same distances between concentric orbits.



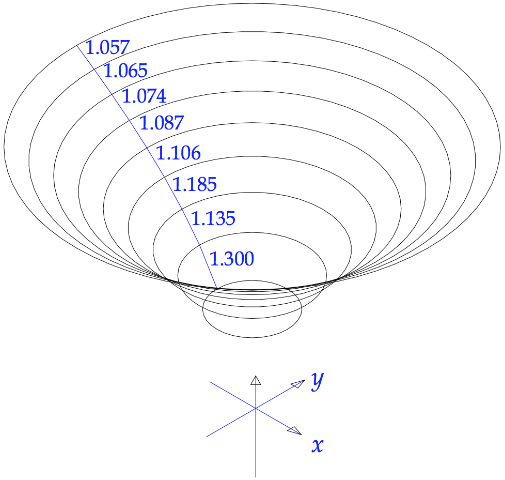
# Circular orbits in space warped by a black hole

- Imagine a BH whose horizon has circumference $2\pi$.
- The distant observer still reports distances of 1 between each orbit.
- The local observer reports warped distances.
- **Both observers measure the same circumferences!**



# Visualization of warped space: "Hyperspace"

- To connect these circles with segments of the "too long" lengths, it is helpful to consider them to be offset from each other along an **imaginary** dimension ⊥ to $x$ and $y$ but which is not $z$.
- If the additional dimension were $z$, then the circles would not appear to lie on a plane.
- Such additional dimensions comprise **hyperspace**.




Hyperspace

# Embedding diagrams

- This is why you often see the equatorial plane of a black hole represented as a funnel-shaped surface, as if made from a stretched rubber sheet.
- It is important to note that the direction of stretch is in hyperspace.
- The scene would **not** look like a funnel to an observer of three spatial dimensions.