Astronomy 142 — Recitation 12

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Formulas to remember

Galaxy distribution in clusters

Number per unit area on the sky

$$\mathcal{N}(r) = \mathcal{N}_0 e^{-(r/r_0)^{-1/4}} \tag{1}$$

Relaxation time

$$t_c \cong \left(\frac{2r_0}{v}\right) \frac{N}{24 \ln\left(\frac{N}{2}\right)} \tag{2}$$

Thermal and escape speeds

$$v_{\text{th}} = \sqrt{\frac{3kT}{m_{\text{H}}}} \qquad v_{\text{esc}} = \sqrt{\frac{2GM}{r_0}} \tag{3}$$

Virial mass

Of a thermalized cluster

$$M = \frac{2r_0\overline{v^2}}{G} = \frac{6r_0\overline{v_r^2}}{G} \tag{4}$$

Mass from Keplerian orbits

$$M(r) = \frac{rv^2}{G} \tag{5}$$

Observed temperature of a distant blackbody

$$T_0 = T\frac{R(t)}{R(t_0)} = \frac{T}{1+z} \tag{6}$$

Robertson-Walker absolute interval

$$ds^2 = c^2dt^2 - R^2(t)\left[\frac{dr_*^2}{1-kr_*^2} + r_*^2 d\theta^2 + r_*^2 \sin^2\theta d\phi^2\right] \tag{7}$$

Scale factor

$$Rr_* = r \qquad R_0 = R(t_0) \quad \text{(today)} \tag{8}$$

Normalized scale factor

$$a(t) = \frac{R(t)}{R_0} = \frac{1}{1+z} \tag{9}$$

Friedmann equation

Neglecting radiation, so only good to $z \sim 3600\ldots$

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8\pi G}{3}\rho_m - \frac{\Lambda}{3} = -k\frac{c^2}{R^2} \tag{10}$$

$$\dot{a}^2 = H_0^2\left[1 + \Omega_{M_0}\left(\frac{1}{a} - 1\right) + \Omega_{\Lambda_0}\left(a^2 - 1\right)\right] \tag{11}$$

Critical density

(present value)

$$u_{c0} = \frac{3c^2H_0^2}{8\pi G} = 9.3 \times 10^{-9}\ \text{erg/cm}^3 \tag{12}$$

Normalized densities, today

Neglecting radiation, to good approximation

$$\Omega_{M0} = \frac{8\pi G\rho_{M0}}{3H_0^2} \qquad \Omega_{\Lambda 0} = \frac{\Lambda}{3H_0^2} \qquad k = \frac{H_0^2R_0^2}{c^2}\left(\Omega_{M0} + \Omega_{\Lambda 0} + 1\right) \tag{13}$$

Workshop problems

Remember! The workshop problems that you will do in groups in Recitation are a crucial part of the process of building up your command of the concepts important in ASTR 142 and subsequent courses. Do not, therefore, do your work on scratch paper and discard it. Better for each of you to keep your own account of each problem in some sort of bound notebook.

1. The Coma Cluster contains about 10,000 galaxies, has a core radius r_0 of about 3 Mpc, and has a radial-velocity dispersion $\sqrt{\overline{v_r^2}}$ of 977 km/s. Typically, the luminosity of the cluster galaxies is $L_g = 5 \times 10^8 L_\odot$. The cluster also contains $M_{\text{X-ray}} = 3 \times 10^{14} M_\odot$ in gas at temperature $T = 10^8$ K.
   (a) Calculate the mass in galaxies (using the results of Problem 2 from Recitation #11) and the virial mass for the Coma Cluster. Compare these results to the mass of X-ray-emitting gas. What fraction of the cluster's mass is dark matter?
   (b) Calculate the crossing time and the relaxation time for the cluster, and compare these with the age of the Universe (14 billion years). Has the cluster had time to come to virial equilibrium in the usual manner, by numerous elastic collisions of the galaxies?
   (c) Calculate the thermal speed of hydrogen in the hot X-ray-emitting gas, the escape speed from the sum of the galaxy and X-ray masses, and the escape speed from the virial mass. Is the presence of hot X-ray-emitting gas a strong argument for the presence of substantial dark matter on its own?

2. So far, we know of three different forms of the absolute interval between events: that for flat spacetime (special relativity), which in Cartesian form is

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

For the surroundings of the horizon of a nonspinning black hole of mass M, the absolute interval in the spherical coordinates of a distant observer is

$$ds^2 = \left(1 - \frac{2GM}{c^2}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{2GM}{rc^2}} - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

The absolute interval that corresponds to the Robertson-Walker metric in stretchy-spherical-grid coordinates is

$$ds^2 = c^2 dt^2 - R^2 \left(\frac{dr_*^2}{1 - kr_*^2} + r_*^2 d\theta^2 + r_*^2 \sin^2\theta d\phi^2\right)$$

(a) Consider two events consisting of ticks on a single given clock. What is the algebraic sign of ds^2 between these two events? (This would be called a timelike interval.)

(b) Consider two events consisting of positions in space with separation measured by a ruler. What is the algebraic sign of ds^2 between these two events? (This would be called a spacelike interval.)

(c) The form of the absolute interval in the vicinity of a mass-density singularity is very complicated, but it has one particularly striking property: $ds^2 < 0$ under all conditions. Is there such a thing as a timelike interval near a singularity? Is there such a thing as time (as we know it) near a mass-density singularity?

(d) What happened before the Big Bang?

3. Using special relativity, demonstrate that light moves along paths with $ds^2 = 0$. (Such paths are called **null geodesics**.)

4. The highest redshift observed to date for a quasar is $z = 7.54$.

   (a) How long ago, in years, did the quasar emit the light that we detect today? So how many light years distant is the quasar? Express your answer in Mpc. This distance is called the **light-travel-time distance**.

   (b) How long, in Mpc, is the null geodesic connecting here and now with the quasar at the time that the light was emitted? This distance is generally known as the **comoving radial distance**. We calculated the comoving radial distance to the decoupling surface, Δr, yesterday in lecture, in the course of measuring the curvature of the universe, k.

   (c) We know of galaxies more distant than this, currently out to a redshift $z = 13.27$. Why is it so special to find such a high redshift quasar?

Intro to Python (A feature *exclusive* of ASTR 142 recitations.)

4. **A closed universe**: Consider a universe with $\Omega_{m0} = \Omega > 1$ and $\Omega_\Lambda = 0$.

   (a) In a procedure similar to that followed in class for the $\Omega \in [0, 1]$ flat universe, solve the Friedmann equation to obtain a relation between the normalized scale factor a and time since the Big Bang t.

   (b) Plot a against t and calculate the age of this universe in Gyr for $\Omega_{m0} = 2$ and our usual value of H_0.

   (c) This is a closed universe, so the expansion eventually halts, reverses, and collapses to a point (generally called a "Big Crunch"). The collapse and Big Crunch are not evident in the formula or plot in parts a and b. Take a guess as to why not.