GR & the Universe

Robertson-Walker Metric
Friedmann Equation
Ages and Fates of Flat Universes
The Cosmological Constant

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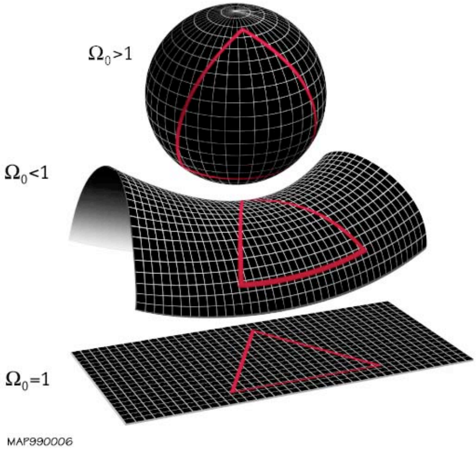
General Relativity & the Universe

- The Robertson-Walker metric and its use
- The Friedmann equation and its solutions
- The ages and fates of flat universes
- The cosmological constant

Reading: Kutner Sec. 20.4, Ryden Sec. 23.2–23.5



Possible spatial geometries of the Universe: from Wikimedia Commons and G. Hinshaw (NASA).

Questions about the Universe

We now know that the Universe is:

- **Expanding**, **isotropic** (uniform in all directions), and **homogeneous** (smooth, similar on large scales)
- Ruled by gravity and the kinetic energy of galaxies and clusters

But this raises several questions:

- How can we use the observations of galaxy motions and distributions, along with our theories of gravity, to determine the density of mass and energy on large scales?
- What is the age of the Universe?
- What is the fate of the Universe? How will the expansion change with time?

A Newtonian universe?

In an infinite, **homogeneous**, and **isotropic** Universe, the gravitational acceleration g should be zero everywhere. Why?

Averaged over large enough scales, a test mass should be equally attracted to all directions in a homogeneous and isotropic universe.

If space is Euclidean and gravity is Newtonian, the only way to get this property is if **space is completely empty**. Reason:

- A large sphere can be drawn through any point.
- The acceleration g at this point due to matter outside the sphere is zero (that matter is uniform, infinite,...).
- The acceleration g due to matter within the sphere is

$$g = \frac{GM(r)}{r^2}$$

  and so $g = 0$ iff the sphere is empty — $M(r) = 0$ — and all other such spheres are empty, by homogeneity.

General Relativistic universes

The Universe is not empty, so it cannot be Newtonian. Fortunately, General Relativity offers a good description, and large-scale structure was the first problem to which Einstein applied GR.

- Note: GR and the solutions to the **Einstein field equations**

$$G_{\mu\nu} + g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4} T_{\mu\nu}$$

  are covered in PHYS 231 and upper-level physics courses.
- The field equations are nonlinear second-order differential equations involving symmetric second-rank tensors. They can be solved uniquely, like any other differential equation, given enough boundary conditions.
- The solution to Einstein's equations is a second-rank tensor called the **metric** ($g_{\mu,\nu}$ above), which describes the measures and curvatures of spacetime under the given boundary conditions.

General Relativistic universes

For any given metric, we can define an **absolute interval** between two events in spacetime.

Flat spacetime

In flat spacetime with no mass present — the limit where GR converges to special relativity — the infinitesimal absolute interval is

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

where dx, dy, dz, and dt are infinitesimal intervals between two events, measured in one reference frame.

The combination ds^2 is independent of reference frame, as you will show in the homework.

The flat-space interval is sometimes called the **Minkowski interval**.

General Relativistic universes

Curved spacetime outside a spherical mass M

The **Schwarzschild metric** describes the solution to Einstein's equations outside a spherical mass M with zero charge and angular momentum (and $\Lambda = 0$). Its absolute interval, in spherical coordinates, is

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2dt^2 - \frac{dr^2}{1 - \frac{2GM}{rc^2}} - r^2d\theta^2 - r^2\sin^2\theta d\phi^2$$

where $r > \frac{2GM}{c^2}$.

The expressions for the spacetime structure outside a black hole, discussed when we covered solar-mass black holes, were derived from this absolute interval.

General Relativistic universes

Isotropic and homogeneous universe

The solution to Einstein's equations in an expanding, isotropic, and homogeneous Universe are called the **Robertson-Walker metric**, which corresponds to an absolute interval

$$ds^2 = c^2 dt^2 - R(t)^2 \left[\frac{dr_*^2}{1 - kr_*^2} + r_*^2 d\theta^2 + r_*^2 \sin^2 \theta d\phi^2 \right]$$

where

$R(t)$	is the **scale factor**: radius of curvature of the Universe
$k = 0, \pm 1$	depending on the curvature
r_*, θ, ϕ	are spherical "comoving" coordinates (dimensionless)

General Relativistic universes

The scale factor R which appears in the Robertson-Walker interval is the solution to the modified Friedmann equation, one component of Einstein's equations for homogeneity/isotropy:

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8\pi G}{3}\rho_M - \frac{\Lambda}{3} = -k\frac{c^2}{R^2}$$

where

ρ_M is the mass density

Λ is the **Cosmological constant**. Not originally part of GR; placed in *ad hoc* by Einstein to permit the field equations to have static (time-independent) solutions.

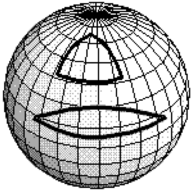
$\dot{R} = \frac{\partial R}{\partial t}$ where $\dot{R} = 0$ for $\Lambda = \frac{3kc^2}{R^2} - 8\pi G\rho_M$

General Relativistic universes

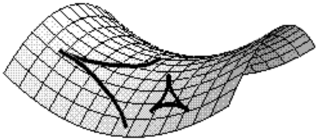
Comparing measurements of galaxy motions and distributions ($\dot{R}$, Rr_*) with solutions of these equations can be used to determine ρ_M, Λ, and k.

- k is the sign of space curvature, *not spacetime*.
- 2D examples: $k = 1$ is a spherical surface; $k = 0$ is a flat plane; $k = -1$ is a hyperboloidal surface.
- See discussion in Ryden Ch. 23.3.

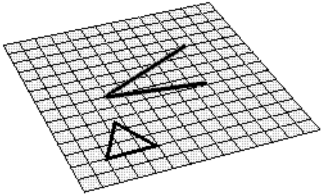
From Nick Strobel's Astronomy Notes.



Universe with *positive* curvature. Diverging line converge at great distances. Triangle angles add to more than 180°.



Universe with *negative* curvature. Lines diverge at ever increasing angles. Triangle angles add to less than 180°.



Universe with no curvature. Lines diverge at constant angle. Triangle angles add to 180°.

Other symbols you may see

You might occasionally encounter other versions of these equations using slightly different symbols and definitions:

Other	**Here**
$d\ell^2$	$-ds^2$
$dx^2 + dy^2 + dz^2$	$-d\ell^2$
$a(t)r_{c,0}$	$R(t)$
a	a
κ	k

Some others: Scale factor $a(t)$ is dimensionless, comoving radius r has dimensions of length.

Here: Scale factor $R(t)$ has dimensions of length, comoving radial coordinate r_* is dimensionless.

How to use the R-W interval

Example: Proper distance

Calculate the distance between two galaxies at some time t — i.e., for $dt = 0$ — choosing both to lie along the x axis, so that $\theta = \phi = d\theta = d\phi = 0$.

$$ds^2 = c^2 dt^2 - d\ell^2 = -d\ell^2 = -R(t)^2 \frac{dr_*^2}{1 - kr_*^2}$$

$$\ell = \int_0^{r_*} d\ell = R(t) \int_0^{r_*} \frac{dr'_*}{\sqrt{1 - kr'^2_*}} = \begin{cases} R(t) \sin^{-1} r_* & k = +1 \\ R(t) r_* & k = 0 \\ R(t) \sinh^{-1} r_* & k = -1 \end{cases}$$

The dimensionless radial coordinate r_* is related to distance in an intuitive way if $k = 0$: just by the scale factor $R(t)$ at time t.

How to use the R-W interval

Example 2: Expansion speed

Calculate the expansion speed if $r_* \ll 1$ (viewing a nearby galaxy).

$$\arcsin r_* = r_* + \frac{1}{6}r_*^3 + \frac{3}{40}r_*^5 + \ldots \approx r_*$$
$$\operatorname{arcsinh} r_* = r_* - \frac{1}{6}r_*^3 + \frac{3}{40}r_*^5 - \ldots \approx r_*$$

so $\ell = R(t)r_*$ for all curvatures. Thus

$$v_r = v = \frac{d\ell}{dt} = \frac{dR}{dt}r_* = \dot{R}(t)r_* = \frac{\dot{R}(t)}{R(t)}\ell = H(t)\ell$$

This last result is just Hubble's Law. Our value of the Hubble "constant," $H_0 = 73.04$ km/s/Mpc, is the present value of $\frac{\dot{R}(t)}{R(t)}$.

How to use the R-W interval

Example 3: Scale factor and redshift

Small distances $r_* \ll 1$ as functions of time:

$$\ell = R(t) r_* \implies \frac{\ell_1}{\ell_0} = \frac{R(t_1)}{R(t_0)}$$

This works for wavelengths too, which are quite small distances. Suppose light is emitted at t_0 and detected at time t_1; then its wavelengths at those two epochs are related by

$$\frac{\lambda_0}{\lambda_1} = \frac{R(t_0)}{R(t_1)}$$

but $\frac{\lambda_0 - \lambda_1}{\lambda_1} = z$, so

$$1 + z = \frac{R(t_0)}{R(t_1)}$$

How to use the R-W interval

Example 4: Critical density

Combine the source terms and use the R-W result from before:

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8\pi G}{3}\left(\rho_M + \frac{\Lambda}{8\pi G}\right) = -k\frac{c^2}{R^2}$$
$$H^2 - \frac{8\pi G}{3c^2}u = -k\frac{c^2}{R^2}$$

Suppose a universe were exactly flat ($k = 0$), described by Euclidean geometry. That would correspond to a special value of the energy density (a critical density) at each time t:

$$H^2 - \frac{8\pi G}{3c^2}u_c = 0 \implies \boxed{u_c = \frac{3c^2 H^2}{8\pi G}}$$

How to use the R-W interval

Example 4: Critical density

The critical density has a **current** value in our Universe of

$$u_{c,0} = \frac{3c^2H_0^2}{8\pi G} = 9.06 \times 10^{-9}\ \mathrm{erg/cm^3}$$

It turns out that our Universe is very nearly flat, so it is customary to define **normalized energy densities** in terms of the critical density:

$$\Omega = \frac{u}{u_c} \qquad \Omega_M = \frac{\rho_M c^2}{u_c} = \frac{8\pi G \rho_M}{3H^2} \qquad \Omega_\Lambda = \frac{\Lambda c^2}{8\pi G u_c} = \frac{\Lambda}{3H^2}$$

In a flat universe,

$$\Omega = \Omega_M + \Omega_\Lambda = 1$$

How to use the Friedmann Equation: The constants

We can express the Friedmann equation in a simpler form in terms of the present-day normalized densities by noting a few things about the constants it contains.

Cosmological constant Λ At present,

$$\frac{\Lambda}{3} = \frac{3H^2\Omega_\Lambda}{3} = H_0^2\Omega_{\Lambda_0}$$

Mass density ρ_M Since a universe stays homogeneous and isotropic as it expands, the mass contained within a sphere of radius R is constant ($\rho_M R^3 = \rho_{M_0} R_0^3$), so

$$\frac{8\pi G}{3}\rho_M = \frac{8\pi G}{\rho_{M_0}}\frac{R_0^3}{R^3} = \frac{8\pi G}{3}\frac{3H_0^2\Omega_{M_0}}{3\pi G}\frac{R_0^3}{R^3} = H_0^2\Omega_{M_0}\frac{R_0^3}{R^3}$$

How to use the Friedmann Equation: The constants

Curvature k is a constant, so we can evaluate it from the Friedmann Eqn. written for the present time:

$$H_0^2 - H_0^2 \Omega_{M_0} \frac{R_0^3}{R^3} - H_0^2 \Omega_{\Lambda_0} = -k \frac{c^2}{R_0^2}$$

$$\therefore k = \frac{H_0^2 R_0^2}{c^2} (\Omega_{M_0} + \Omega_{\Lambda_0} - 1)$$

Since $\frac{H_0^2 R_0^2}{c^2}$ is positive definite, the sign of k is determined by the sum of the normalized densities: The universe is positively curved if $\Omega_{M_0} + \Omega_{\Lambda_0} > 1$, and negatively curved if $\Omega_{M_0} + \Omega_{\Lambda_0} < 1$.