# Finding the Universe among the universes

Friedmann Equation Ages and Fates of Flat Universes The Cosmological Constant Constraining the constants in the Friedmann Equation

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# Finding the Universe among the universes

- The Friedmann equation and its solutions
- The ages and fates of flat universes
- The cosmological constant
- Constraining the constants in the Friedmann equation:
  - Ages of globular clusters
  - Galaxy distributions and  $\Omega_{M_0}$
  - Acceleration of high-redshift galaxies

Reading: Kutner Sec. 21.4, Ryden Sec. 24.1-24.2

*Image of the WMAP spacecraft, which operated from 2001 to 2010.* 



#### How to use the Friedmann Equation: The constants Put all these terms back into the Friedmann Eqn. and multiply through by $R^2$ :

$$\begin{pmatrix} \frac{\dot{R}}{R} \end{pmatrix}^2 - \frac{8\pi G}{3} \rho_M - \frac{\Lambda}{3} = -k \frac{c^2}{R^2} \dot{R}^2 - H_0^2 \Omega_{M_0} \frac{R_0^3}{R} - H_0^2 \Omega_{\Lambda_0} R^2 = -H_0^2 R_0^2 (\Omega_{M_0} + \Omega_{\Lambda_0} - 1) \left( \frac{\dot{R}}{R_0} \right)^2 = H_0^2 \left[ 1 + \Omega_{M_0} \left( \frac{R_0}{R} - 1 \right) + \Omega_{\Lambda_0} \left( \frac{R^2}{R_0^2} - 1 \right) \right]$$

Defining the **normalized scale factor**  $a = R/R_0$ , with a = 1 today, we can rewrite the Friedmann equation:

$$\dot{a}^{2} = H_{0}^{2} \left[ 1 + \Omega_{M_{0}} \left( \frac{1}{a} - 1 \right) + \Omega_{\Lambda_{0}} \left( a^{2} - 1 \right) \right]$$

The Friedmann Equation is separable and directly integrable.

Example: A flat universe

Suppose a universe were flat, i.e.,

$$\Omega \in [0,1]$$
  $\Omega_{M_0} = \Omega$   $\Omega_{\Lambda_0} = 1 - \Omega$ 

What is the relation between time and normalized scale factor *a*?

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$$\begin{aligned} \left(\frac{da}{dt}\right)^2 &= H_0^2 \left[1 + \Omega\left(\frac{1}{a} - 1\right) + (1 - \Omega)\left(a^2 - 1\right)\right] \\ &= H_0^2 \left[1 + \frac{\Omega}{a} - \Omega + a^2 - 1 - \Omega a^2 + \Omega\right] \\ &= H_0^2 \left(\frac{\Omega}{a} + a^2 - \Omega a^2\right) = \frac{H_0^2 \Omega}{a} \left(1 + \frac{1 - \Omega}{\Omega} a^3\right) \end{aligned}$$

#### Example: A flat universe

#### Make the following substitutions:

$$x = \left(\frac{1-\Omega}{\Omega}\right)^{1/3} a \qquad \qquad \frac{dx}{da} = \left(\frac{1-\Omega}{\Omega}\right)^{1/3}$$

so that  $x \in [0, ((1 - \Omega) / \Omega)^{1/3} a]$  as  $a' \in [0, a]$ .

Multiply through by  $(dx/da)^2$  and use the chain rule:

$$\left(\frac{dx}{da}\frac{da}{dt}\right)^2 = \left(\frac{1-\Omega}{\Omega}\right)^{2/3}\frac{H_0^2\Omega}{a}(1+x^3)$$
$$\left(\frac{dx}{dt}\right)^2 = H_0^2(1-\Omega)\frac{1+x^3}{x}$$

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#### Example: A flat universe

Now take the square root, separate, and integrate. It is easier to work with t(x) rather than x(t):

$$t(x) = \int_0^t dt' = \frac{1}{H_0\sqrt{1-\Omega}} \int_0^{((1-\Omega)/\Omega)^{1/3}a} dx \sqrt{\frac{x}{1+x^3}}$$

This integral can be done analytically (see slides at end):

$$t(a) = \frac{2}{3H_0\sqrt{1-\Omega}} \ln\left(\sqrt{a^3 \frac{1-\Omega}{\Omega}} + \sqrt{1+a^3 \frac{1-\Omega}{\Omega}}\right)$$

#### Example: A flat universe

Let us plot a(t) against t and define the age and fate of flat universes: Age since the Big Bang (a = 0) is the time

from the present: (u = 0)

$\Omega_{M_0}$	Age (Gyr)
0.25	13.6
0.50	11.1
0.75	9.81
1.00	8.93

The first of these universes has an age not far from that of an empty universe,  $t = 1/H_0 = 13.4$  Gyr.



#### Example: A flat universe

**Fate**: No flat universes with  $\Omega_{\Lambda_0} \ge 0$  will collapse.

All of the universes expand exponentially (are **open**), except for the pure-matter flat universe ( $\Omega_{\Lambda_0} = 0$ ,  $\Omega_{M_0} = 1$ ).

In the pure-matter flat universe, the expansion continues forever but not as fast  $(a \propto t^{2/3})$ .



#### Notes on the cosmological constant

Despite the appearance that it functions as just another energy density, the cosmological constant  $\Lambda$  is very different from the density terms.

- The total mass of a physical universe is constant energy is conserved so the mass density decreases monotonically with time in an expanding universe.
- But Λ, being constant, *does not decrease with time*. As the volume of an expanding universe increases, the total energy represented by Λ, which we call **dark energy**, increases.
- The dark energy in a universe is not necessarily conserved. Its increase is responsible for the exponential, accelerating expansion in flat universes with nonzero  $\Omega_{\Lambda}$ .

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## Solving the integral...

In order to solve the integral

$$t = \int_0^{((1-\Omega)/\Omega)^{1/3}a} dx \sqrt{\frac{x}{1+x^3}}$$

first make the substitution

$$u = x^{3/2}$$
  $du = \frac{3}{2}x^{1/2}dx$   $u \in \left[0, \left(\frac{1-\Omega}{\Omega}\right)^{1/2}a^{3/2}\right]$ 

Thus

$$t = \frac{2}{3} \int_0^{((1-\Omega)/\Omega)^{1/2} a^{3/2}} \frac{du}{\sqrt{1+u^2}}$$

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## Solving the integral...

This is equal to

$$t = \frac{2}{3} \sinh^{-1} u \Big|_{0}^{((1-\Omega)/\Omega)^{1/2} a^{3/2}}$$
$$t = \frac{2}{3} \sinh^{-1} \left( \sqrt{\frac{1-\Omega}{\Omega}} a^{3/2} \right)$$

Remembering that  $\sinh^{-1} u = \ln \left( u + \sqrt{u^2 + 1} \right)$ ,

$$t = \frac{2}{3} \ln \left( \sqrt{a^3 \frac{1 - \Omega}{\Omega}} + \sqrt{a^3 \left( \frac{1 - \Omega}{\Omega} \right) + 1} \right)$$

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#### Is our Universe among these choices?

$$\dot{a}^2 = H_0^2 \left[ 1 + \Omega_{M_0} \left( rac{1}{a} - 1 
ight) + \Omega_{\Lambda_0} \left( a^2 - 1 
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A wide range of ages and fates is consistent with homogeneity, isotropy, and expansion at 73.04 km/s/Mpc. So we need more experimental constraints:

- Independent measurements of the age of the Universe. For example, the age of the Milky Way's globular clusters, derived from main-sequence turnoff, is ~ 13 Gyr (Krauss & Chaboyer 2003, Marin et al. 2009). This sets a lower bound on the age of the Universe, which must be at least a bit older than its contents.
- Measurements of the **mass density**  $\Omega_M$ .
- Measurements of the acceleration or deceleration, *a*, of the expansion of the Universe.
- Measurements of the **curvature**, *k*, of the Universe.

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# Measurements of the mass density

One way to determine the mass density,  $\rho_{M_0}$  or  $\Omega_{M_0}$ , is to measure fluctuations in the 3-D distributions of galaxies or galaxy clusters determined from sky surveys and radial velocity measurements.

- Principle: relate statistical variation in galaxy distribution to average values.
- Statistical variation is measured by galaxy or cluster correlation functions, or their corresponding power spectra.
- Compare to statistics of gravitational structure formation models to get the total average density, including dark matter.



# Measurements of the mass density

Observational bounds on  $\Omega_M$  made from the number density of galaxy clusters (Planck Collaboration 2018) and hence sensitive to luminous and dark matter:

 $\Omega_M = 0.3111 \pm 0.0056$ 

The clusters were observed in measurements by the Planck satellite using the Sunyaev-Zel'dovich effect (Sunyaev & Zel'dovich 1970).



The contours describe different methods of measuring the parameters; the blue ellipses show fit results from 2018 (Planck Collaboration 2018). Note that  $\sigma_8$  is the measured RMS in galaxy numbers within spheres of radius 8  $h^{-1}$  Mpc, where  $H_0 = 100h$  km s<sup>-1</sup> Mpc<sup>-1</sup>.

## Measurements of the mass density

Globular cluster ages and measurements of  $\Omega_{M_0}$  firmly rule out the possibility that we are in a matter-dominated universe, but universes with a nonzero cosmological constant appear to solve the age problem fairly easily.

The expansion of the universe decelerates for large  $\Omega_{M_0}$  due to the effect of gravity.

But, if  $\Omega_{\Lambda_0} \neq 0$ , the accumulation of dark energy eventually accelerates the expansion. So, look for observational evidence of acceleration.



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In our usual plots of *a* and *t*, **deceleration** appears as concave-downward curvature, as in the universes with  $\Lambda = 0$ .

And **acceleration** appears as concave-upward curvature, as in our results for flat universes several Gyr after the Big Bang.

So we should look for observational evidence of  $\Lambda \neq 0$ .



Recall that the normalized scale factor *a* is proportional to the typical distance between galaxies in the Universal expansion.

Thus acceleration and deceleration appear as departures from the Hubble diagram, which is linear in this rendition.

- Above the empty universe: objects will appear fainter than expected from their distances.
- Below the empty universe: objects will appear brighter than expected.

Note how the "empty" *a*-*t* plot has slope  $H_0$  and *t*-intercept  $1/H_0$ .



Using SNe Ia found in  $z \sim 0.1 - 1$ , evidence was found for **acceleration** (Knop et al. 2003). Note the presence of systematic uncertainty due to lower metallicity in the distant past.

**Big caveat:** The distances to these galaxies are still, of course, measured by SNe Ia luminosity.

- Their light was emitted far enough in the past that metal abundances were substantially smaller than today.
- So we probably do not know the SN Ia luminosity of these objects accurately enough to confidently use them as standard candles at high redshift.

SNe Ia observations in 1998 from the Supernova Search Team (Riess et al. 1998) .



Despite the considerable and ongoing uncertainties about high-redshift SNe Ia, the observation of Universal acceleration was the first respectable indication of a non-zero cosmological constant.

The 2011 Nobel Prize in Physics went to Saul Perlmutter, Adam Riess, and Brian Schmidt, who led the SNe Ia surveys.



## Summary: The Universe today



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