Finding the Universe among the universes

Friedmann Equation
Ages and Fates of Flat Universes
The Cosmological Constant
Constraining the constants in the Friedmann Equation

April 26, 2024

University of Rochester

Finding the Universe among the possible universes

- The Friedmann equation and its solutions
- The ages and fates of flat universes
- The cosmological constant
- Constraining the constants in the Friedmann equation:
  - Ages of globular clusters
  - Galaxy distributions and Ω_{M_0}
  - Acceleration of high-redshift galaxies

Reading: Kutner Sec. 21.4, Ryden Sec. 24.1–24.2

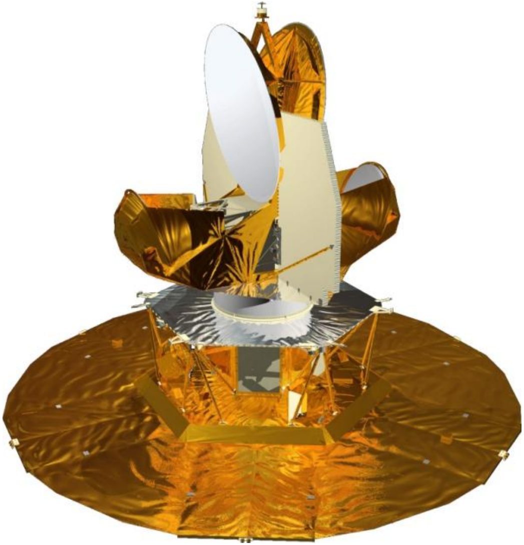


Image of the WMAP spacecraft, which operated from 2001 to 2010.

General Relativistic universes

The scale factor R which appears in the Robertson-Walker interval is the solution to the modified Friedmann equation, one component of Einstein's equations for homogeneity/isotropy:

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8\pi G}{3}\rho_M - \frac{\Lambda}{3} = -k\frac{c^2}{R^2}$$

where

ρ_M is the mass density

Λ is the **Cosmological constant**. Not originally part of GR; placed in *ad hoc* by Einstein to permit the field equations to have static (time-independent) solutions.

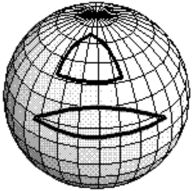
$\dot{R} = \frac{\partial R}{\partial t}$ where $\dot{R} = 0$ for $\Lambda = \frac{3kc^2}{R^2} - 8\pi G\rho_M$

General Relativistic universes

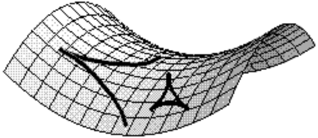
Comparing measurements of galaxy motions and distributions ($\dot{R}$, Rr_*) with solutions of these equations can be used to determine ρ_M, Λ, and k.

- k is the sign of space curvature, *not spacetime*.
- 2D examples: $k = 1$ is a spherical surface; $k = 0$ is a flat plane; $k = -1$ is a hyperboloidal surface.
- See discussion in Ryden Ch. 23.3.

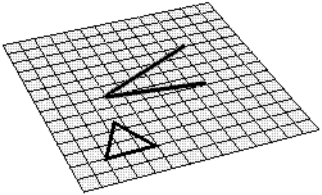
From Nick Strobel's Astronomy Notes.



Universe with *positive* curvature. Diverging line converge at great distances. Triangle angles add to more than 180°.



Universe with *negative* curvature. Lines diverge at ever increasing angles. Triangle angles add to less than 180°.



Universe with no curvature. Lines diverge at constant angle. Triangle angles add to 180°.

Other symbols you may see

You might occasionally encounter other versions of these equations using slightly different symbols and definitions:

Other	**Here**
$d\ell^2$	$-ds^2$
$dx^2 + dy^2 + dz^2$	$-d\ell^2$
$a(t)r_{c,0}$	$R(t)$
a	a
κ	k

Some others: Scale factor $a(t)$ is dimensionless, comoving radius r has dimensions of length.

Here: Scale factor $R(t)$ has dimensions of length, comoving radial coordinate r_* is dimensionless.

How to use the R-W interval

Example: Proper distance

Calculate the distance between two galaxies at some time t — i.e., for $dt = 0$ — choosing both to lie along the x axis, so that $\theta = \phi = d\theta = d\phi = 0$.

$$ds^2 = c^2 dt^2 - d\ell^2 = -d\ell^2 = -R(t)^2 \frac{dr_*^2}{1 - kr_*^2}$$

$$\ell = \int_0^{r_*} d\ell = R(t) \int_0^{r_*} \frac{dr'_*}{\sqrt{1 - kr'^2_*}} = \begin{cases} R(t) \sin^{-1} r_* & k = +1 \\ R(t) r_* & k = 0 \\ R(t) \sinh^{-1} r_* & k = -1 \end{cases}$$

The dimensionless radial coordinate r_* is related to distance in an intuitive way if $k = 0$: just by the scale factor $R(t)$ at time t.

How to use the R-W interval

Example 2: Expansion speed

Calculate the expansion speed if $r_* \ll 1$ (viewing a nearby galaxy).

$$\arcsin r_* = r_* + \frac{1}{6}r_*^3 + \frac{3}{40}r_*^5 + \ldots \approx r_*$$
$$\operatorname{arcsinh} r_* = r_* - \frac{1}{6}r_*^3 + \frac{3}{40}r_*^5 - \ldots \approx r_*$$

so $\ell = R(t)r_*$ for all curvatures. Thus

$$v_r = v = \frac{d\ell}{dt} = \frac{dR}{dt}r_* = \dot{R}(t)r_* = \frac{\dot{R}(t)}{R(t)}\ell = H(t)\ell$$

This last result is just Hubble's Law. Our value of the Hubble "constant," $H_0 = 73.04$ km/s/Mpc, is the present value of $\frac{\dot{R}(t)}{R(t)}$.

How to use the R-W interval

Example 3: Scale factor and redshift

Small distances $r_* \ll 1$ as functions of time:

$$\ell = R(t)r_* \implies \frac{\ell_1}{\ell_0} = \frac{R(t_1)}{R(t_0)}$$

This works for wavelengths too, which are quite small distances. Suppose light is emitted at t_0 and detected at time t_1; then its wavelengths at those two epochs are related by

$$\frac{\lambda_0}{\lambda_1} = \frac{R(t_0)}{R(t_1)}$$

but $\frac{\lambda_0 - \lambda_1}{\lambda_1} = z$, so

$$1 + z = \frac{R(t_0)}{R(t_1)}$$

How to use the R-W interval

Example 4: Critical density

Combine the source terms and use the R-W result from before:

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8\pi G}{3}\left(\rho_M + \frac{\Lambda}{8\pi G}\right) = -k\frac{c^2}{R^2}$$

$$H^2 - \frac{8\pi G}{3c^2}u = -k\frac{c^2}{R^2}$$

Suppose a universe were exactly flat ($k = 0$), described by Euclidean geometry. That would correspond to a special value of the energy density (a critical density) at each time t:

$$H^2 - \frac{8\pi G}{3c^2}u_c = 0 \implies \boxed{u_c = \frac{3c^2H^2}{8\pi G}}$$

How to use the R-W interval

Example 4: Critical density

The critical density has a **current** value in our Universe of

$$u_{c,0} = \frac{3c^2H_0^2}{8\pi G} = 9.06 \times 10^{-9}\ \mathrm{erg/cm^3}$$

It turns out that our Universe is very nearly flat, so it is customary to define **normalized energy densities** in terms of the critical density:

$$\Omega = \frac{u}{u_c} \qquad \Omega_M = \frac{\rho_M c^2}{u_c} = \frac{8\pi G\rho_M}{3H^2} \qquad \Omega_\Lambda = \frac{\Lambda c^2}{8\pi G u_c} = \frac{\Lambda}{3H^2}$$

In a flat universe,

$$\Omega = \Omega_M + \Omega_\Lambda = 1$$

How to use the Friedmann Equation: The constants

We can express the Friedmann equation in a simpler form in terms of the present-day normalized densities by noting a few things about the constants it contains.

Cosmological constant Λ At present,

$$\frac{\Lambda}{3} = \frac{3H^2\Omega_\Lambda}{3} = H_0^2\Omega_{\Lambda_0}$$

Mass density ρ_M Since a universe stays homogeneous and isotropic as it expands, the mass contained within a sphere of radius R is constant ($\rho_M R^3 = \rho_{M_0} R_0^3$), so

$$\frac{8\pi G}{3}\rho_M = \frac{8\pi G}{\rho_{M_0}}\frac{R_0^3}{R^3} = \frac{8\pi G}{3}\frac{3H_0^2\Omega_{M_0}}{3\pi G}\frac{R_0^3}{R^3} = H_0^2\Omega_{M_0}\frac{R_0^3}{R^3}$$

How to use the Friedmann Equation: The constants

Curvature k is a constant, so we can evaluate it from the Friedmann Eqn. written for the present time:

$$H_0^2 - H_0^2 \Omega_{M_0} \frac{R_0^3}{R^3} - H_0^2 \Omega_{\Lambda_0} = -k \frac{c^2}{R_0^2}$$

$$\therefore k = \frac{H_0^2 R_0^2}{c^2} (\Omega_{M_0} + \Omega_{\Lambda_0} - 1)$$

Since $\frac{H_0^2 R_0^2}{c^2}$ is positive definite, the sign of k is determined by the sum of the normalized densities: The universe is positively curved if $\Omega_{M_0} + \Omega_{\Lambda_0} > 1$, and negatively curved if $\Omega_{M_0} + \Omega_{\Lambda_0} < 1$.

How to use the Friedmann Equation: The constants

Put all these terms back into the Friedmann Eqn. and multiply through by R^2:

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8\pi G}{3}\rho_M - \frac{\Lambda}{3} = -k\frac{c^2}{R^2}$$

$$\dot{R}^2 - H_0^2\Omega_{M_0}\frac{R_0^3}{R} - H_0^2\Omega_{\Lambda_0}R^2 = -H_0^2R_0^2(\Omega_{M_0} + \Omega_{\Lambda_0} - 1)$$

$$\left(\frac{\dot{R}}{R_0}\right)^2 = H_0^2\left[1 + \Omega_{M_0}\left(\frac{R_0}{R} - 1\right) + \Omega_{\Lambda_0}\left(\frac{R^2}{R_0^2} - 1\right)\right]$$

Defining the **normalized scale factor** $a = R/R_0$, with $a = 1$ today, we can rewrite the Friedmann equation:

$$\boxed{\dot{a}^2 = H_0^2\left[1 + \Omega_{M_0}\left(\frac{1}{a} - 1\right) + \Omega_{\Lambda_0}\left(a^2 - 1\right)\right]}$$

Using the Friedmann Equation

The Friedmann Equation is separable and directly integrable.

Example: A flat universe

Suppose a universe were **flat**, i.e.,

$$\Omega \in [0,1] \qquad \Omega_{M_0} = \Omega \qquad \Omega_{\Lambda_0} = 1 - \Omega$$

What is the relation between time and normalized scale factor a?

$$\begin{aligned}\left(\frac{da}{dt}\right)^2 &= H_0^2\left[1 + \Omega\left(\frac{1}{a} - 1\right) + (1-\Omega)\left(a^2 - 1\right)\right] \\ &= H_0^2\left[1 + \frac{\Omega}{a} - \Omega + a^2 - 1 - \Omega a^2 + \Omega\right] \\ &= H_0^2\left(\frac{\Omega}{a} + a^2 - \Omega a^2\right) = \frac{H_0^2\Omega}{a}\left(1 + \frac{1-\Omega}{\Omega}a^3\right)\end{aligned}$$

Using the Friedmann Equation

Example: A flat universe

Make the following substitutions:

$$x = \left(\frac{1-\Omega}{\Omega}\right)^{1/3} a \qquad \frac{dx}{da} = \left(\frac{1-\Omega}{\Omega}\right)^{1/3}$$

so that $x \in [0, ((1-\Omega)/\Omega)^{1/3} a]$ as $a' \in [0, a]$.

Multiply through by $(dx/da)^2$ and use the chain rule:

$$\left(\frac{dx}{da}\frac{da}{dt}\right)^2 = \left(\frac{1-\Omega}{\Omega}\right)^{2/3} \frac{H_0^2 \Omega}{a}(1+x^3)$$

$$\left(\frac{dx}{dt}\right)^2 = H_0^2 (1-\Omega) \frac{1+x^3}{x}$$

Using the Friedmann Equation

Example: A flat universe

Now take the square root, separate, and integrate. It is easier to work with $t(x)$ rather than $x(t)$:

$$t(x) = \int_0^t dt' = \frac{1}{H_0\sqrt{1-\Omega}} \int_0^{((1-\Omega)/\Omega)^{1/3} a} dx \sqrt{\frac{x}{1+x^3}}$$

This integral can be done analytically (see slides at end):

$$\boxed{t(a) = \frac{2}{3H_0\sqrt{1-\Omega}} \ln\left(\sqrt{a^3\frac{1-\Omega}{\Omega}} + \sqrt{1+a^3\frac{1-\Omega}{\Omega}}\right)}$$

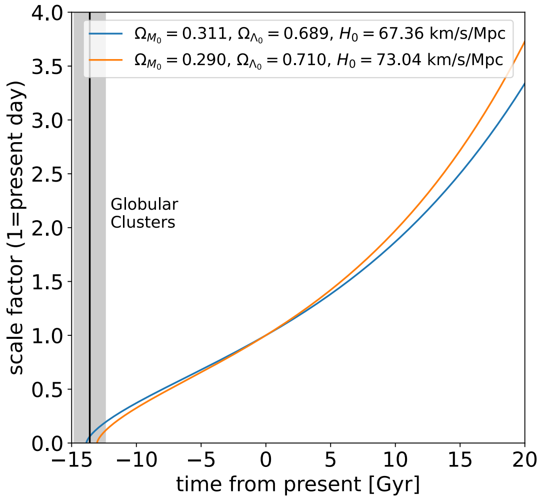
Using the Friedmann Equation

Example: A flat universe

Let us plot $a(t)$ against t and define the age and fate of flat universes:
Age since the Big Bang ($a = 0$) is the time from the present:

Ω_{M_0}	Age (Gyr)
0.25	13.6
0.50	11.1
0.75	9.81
1.00	8.93

The first of these universes has an age not far from that of an empty universe,
$t = 1/H_0 = 13.4$ Gyr.



Using the Friedmann Equation

Example: A flat universe

Fate: No flat universes with $\Omega_{\Lambda_0} \geq 0$ will collapse.

All of the universes expand exponentially (are open), except for the pure-matter flat universe ($\Omega_{\Lambda_0} = 0, \Omega_{M_0} = 1$).

In the pure-matter flat universe, the expansion continues forever but not as fast ($a \propto t^{2/3}$).