Astronomy 142 — Practice Midterm Exam #1

Professor Kelly Douglass

Spring 2025

Name: ______________________________

You may consult *only* one page of formulas and constants and a calculator while taking this test. You may *not* consult any books, digital resources, or each other. All of your work must be written on the attached pages, using the reverse sides if necessary. The final answers, and any formulas you use or derive, must be indicated clearly (answers must be circled or boxed). You will have one hour and fifteen minutes to complete the exam. Good luck!

- First, work on the problems you find the easiest. Come back later to the more difficult or less familiar material. Do not get stuck.
- The amount of space left for each problem is not necessarily an indication of the amount of writing it takes to solve it.
- Numerical answers are incomplete without units and should not be written with more significant figures than they deserve.
- Remember, you can earn partial credit for being on the right track. Be sure to show enough of your reasoning that we can figure out what you are thinking.

$R_\odot = 6.96 \times 10^{10}$ cm	$M_{\rm bol} = 4.74$
$M_\odot = 1.989 \times 10^{33}$ g	$m_V = -26.71$
$L_\odot = 3.827 \times 10^{33}$ erg/s	$M_V = 4.86$
$T_e = 5772$ K	$BC_V = -0.12$
1 AU $= 149,597,870$ km	1 pc $= 206,625$ AU
$k = 1.38 \times 10^{-16}$ erg/K	$\sigma = 5.6704 \times 10^{-5}$ erg s^{-1} cm^{-2} K^{-4}
$G = 6.674 \times 10^{-8}$ dyn cm^2 g^{-2}	$c = 3 \times 10^{10}$ cm/s
$h = 6.6261 \times 10^{-27}$ erg s	$m_p = 1.6726 \times 10^{-24}$ g
$m_n = 1.6749 \times 10^{-24}$ g	$m_e = 9.1094 \times 10^{-28}$ g
$q_e = 4.803 \times 10^{-10}$ esu	

1. **Short answers.** Please write in complete sentences, and feel free to use equations and/or sketches to help explain your thoughts.

   (a) (5 points) The "twin" stars in Gemini, Castor and Pollux, differ in visual magnitude by 0.5. By what factor do they differ in visual flux?

   Solution:

$$m_2 - m_1 = 2.5 \log\left(\frac{f_1}{f_2}\right)$$

$$\frac{f_1}{f_2} = 10^{\Delta m/2.5} = 10^{0.5/2.5}$$

$$\boxed{\frac{f_1}{f_2} = 1.585}$$

   (b) (5 points) Give *two* pieces of evidence or arguments that the Sun is undergoing nuclear reactions in its interior.

   Solution:

   1. Observations of solar neutrinos.
   2. No other energy source can last Gyr timescales; chemical sources last thousands of years, and gravitationally-driven luminosity lasts a few million years at best.

(c) (5 points) Leaving you to watch from a large distance, I assume a stationary position between you and a nonspinning black hole's event horizon and arrange two meter sticks: one pointing directly (radially) away from the black hole, one perpendicular to this direction. Describe the difference in appearance of the meter sticks from your point of view.

> **Solution:** The meter stick pointing radially outwards from the black hole will appear to be shorter than 1 meter in length, while the one perpendicular to this will be unchanged. In the vicinity of the black hole, the meter sticks still appear to be a meter in length to me. Since space appears to be stretched here (a result of the black hole's strong gravitational field), I require more meter sticks to measure the same distance along the radial direction than you. Consequently, it looks to you like my meter sticks are shorter than normal (when aligned in the radial direction).

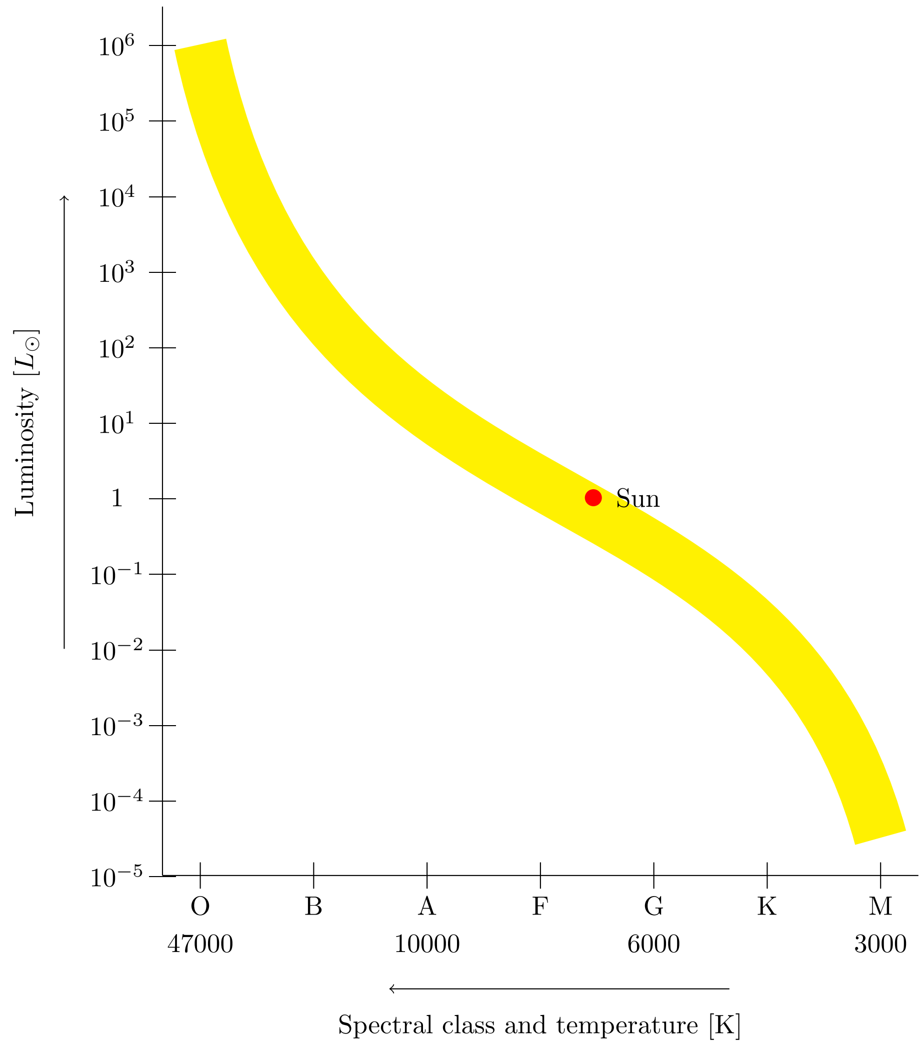
(d) (5 points) Estimate the dates, during the next year, for which the sidereal time at midnight (standard time) is 0h, 6h, 12h, and 18h. What is your reasoning?

> **Solution:** By definition, sidereal time at noon on the vernal equinox is 12h. Midnight on the vernal equinox therefore has a sidereal time of 12h.
>
> Six months later, the sidereal clock will have advanced 12 hours, so the sidereal time at midnight on the autumnal equinox is 0h.
>
> Similarly, the sidereal clock will have advanced 6 hours between the vernal equinox and the summer solstice, so the sidereal time at midnight on the summer solstice is 18h.
>
> And, finally, the sidereal clock will have advanced 6 hours between the autumnal equinox and the winter solstice, so the sidereal time at midnight on the winter solstice is 6h.

2. The HR diagram is the single most useful aid in the characterization of stars.

(a) (10 points) Draw an HR diagram that contains only main sequence stars. Label the axes in physical units. Indicate the location of the O stars, the M stars, and the Sun.

Solution:



(b) (5 points) Where are the most luminous stars on this HR diagram? Where might you find the oldest stars? Are all the stars in this vicinity old? Why or why not? Where are the newest stars for sure?

Solution: The most luminous stars are the O stars — these are located in the upper left corner of the HR diagram. The oldest stars are the M stars, which are located in the bottom right corner of the HR diagram. The low mass of the M stars causes them to burn through their nucleosynthetic fuel at a much slower rate, allowing them to remain in this location on the HR diagram for a very long time. However, not all M stars are old — new, low-mass stars will also be located in this region of the HR diagram. The newest stars are the O type stars: their extremely high mass and luminosity causes them to quickly use up all of their hydrogen and expand into the giant or supergiant phase of their lifespan.

3. **Stellar magnitudes**: Two stars are observed that appear to be companions. They are very close in the sky and both have an annual parallax of 0.1 arcsec. One of the stars is much brighter than the other.

   (a) (10 points) The brighter of the two stars has an apparent bolometric magnitude of 4.74 and its spectrum peaks at 0.5 μm. Estimate the surface temperature and radius of the star, and indicate whether or not it is likely to lie on the main sequence. Express your numerical answers in units of K and $R_\odot$.

Solution: Parallax implies $d = 1 \text{ pc} \cdot 1\text{"}/0.1\text{"} = 10$ pc. Therefore, $m_{\text{bol}} = M_{\text{bol}}$. And since $M_{\text{bol},\odot} = 4.74$, $L = L_\odot$.
From the peak wavelength, we can also get T_e using Wien's Law:

$$T_e = \frac{0.29 \text{ cm K}}{0.5\ \mu\text{m}} = 5800 \text{ K}$$

Given T_e and L, it is easy to find R:

$$\begin{aligned} L &= 4\pi R^2 \sigma T_e^4 \\ R &= \left(\frac{L}{4\pi\sigma T_e^4}\right)^{1/2} \\ &= \left(\frac{3.839 \times 10^{33}}{4\pi \cdot 5.671 \times 10^{-5} \cdot 5800^4}\right)^{1/2} \\ &= 7 \times 10^{10} \text{ cm} = R_\odot \end{aligned}$$

With the same surface temperature and radius as the Sun, this star likely lies on the main sequence of the H–R diagram.

   (b) (10 points) The dimmer companion has an apparent magnitude of 10.69 and its intensity peaks at 0.29 μm. Estimate the surface temperature and radius of the star (in units of K and $R_\odot$, respectively).

Solution: Using the expression relating absolute magnitude M (not to be confused with mass) to luminosity,

$$\begin{aligned} M &= 4.74 - 2.5 \log (L/L_\odot) \\ L &= L_\odot 10^{(4.74-M)/2.5} \\ &= 0.0042 L_\odot \end{aligned}$$

The peak wavelength can be used in Wien's Law to get the effective temperature:

$$T_e = \frac{0.29 \text{ cm K}}{0.29\ \mu\text{m}} = 10^4 \text{ K}$$

Finally, solving for R:

$$\begin{aligned} R &= \left(\frac{L}{4\pi\sigma T_e^4}\right)^{1/2} \\ &= \left(\frac{(0.0042)(3.839 \times 10^{33} \text{ erg/s})}{4\pi(5.6704 \times 10^{-5} \text{ erg s}^{-1} \text{ cm}^{-2} \text{ K}^{-4})(10000 \text{ K})^4}\right)^{1/2} \\ &= 1.5 \times 10^9 \text{ cm} = 0.02 R_\odot \end{aligned}$$

4. **Stellar Pulsations**: A spherical star with radius R has uniform density ρ and is supported by ideal gas pressure.

(a) (10 points) Derive a formula for the pressure P as a function of radius r from the center of the star.

Solution: If ρ is constant, then at a given radius r

$$M(r) = \rho V = \frac{4}{3}\pi\rho r^3$$

Applying hydrostatic equilibrium with the boundary condition $P = 0$ at $r = R$,

$$\begin{aligned}
\frac{dP}{dr} &= -\frac{GM(r)}{r^2}\rho \\
\int_{P(r)}^{0} dP &= -G\rho \int_r^R \frac{M(r')}{r'^2} dr' \\
-P(r) &= -\frac{4}{3}\pi G\rho^2 \int_r^R r'\, dr' \\
&= -\frac{4}{3}\pi G\rho^2 \left[\frac{r'^2}{2}\right]_r^R \\
P(r) &= \frac{2\pi G\rho^2}{3}(R^2 - r^2)
\end{aligned}$$

(b) (5 points) Derive a formula for the temperature T as a function of radius r from the center of the star.

Solution: Use the ideal gas law:

$$\begin{aligned}
P &= nkT = \frac{\rho}{\mu}kT \\
T &= \frac{\mu P}{k\rho} \\
&= \frac{\mu}{k\rho}\frac{2\pi G\rho^2}{3}(R^2 - r^2) \\
T(r) &= \frac{2\pi G\mu\rho}{3k}(R^2 - r^2)
\end{aligned}$$

(c) (15 points) Derive a formula for the period Π of the fundamental mode of radial pulsations for this star. *Hint: Recall that the adiabatic speed of sound is* $v_s = \sqrt{\gamma P/\rho}$. *You may also find the identity* $\int_0^1 du/\sqrt{1-u^2} = \pi/2$ *useful.*

Solution: Boundary conditions imply that 1/4 wavelength fits in the fundamental mode, with the core acting as a node and the surface acting as an antinode. Therefore, the fundamental period is

$$\begin{aligned}
\Pi &= 4\int_0^R \frac{dr}{v_s} \\
&= 4\int_0^R \sqrt{\frac{\rho}{\gamma P(r)}}dr \\
&= 4\int_0^R \sqrt{\frac{3}{\gamma 2\pi G\rho(R^2-r^2)}}dr \\
&= 2\sqrt{\frac{6}{\gamma G\pi\rho}}\int_0^R \frac{dr}{R\sqrt{1-(r/R)^2}}
\end{aligned}$$

Make a change of variables such that

$$u = \frac{r}{R} \qquad du = \frac{dr}{R} \qquad u = 0 \to 1 \text{ as } r = 0 \to R$$

Solving gives

$$\begin{aligned}
\Pi &= 2\sqrt{\frac{6}{\gamma G\pi\rho}}\int_0^1 \frac{du}{\sqrt{1-u^2}} \\
&= 2\sqrt{\frac{6}{\gamma G\pi\rho}}\frac{\pi}{2} \\
&= \sqrt{\frac{6\pi}{\gamma G\rho}}
\end{aligned}$$

5. (15 points) **Double-line eclipsing spectroscopic binary.** Suppose that you can measure the orbital period, P, and the velocity amplitudes, v_1 and v_2, for double-line eclipsing spectroscopic binary stars. Derive formulas for the distance r between the stars, and the masses m_1 and m_2, in terms of these measurements, under the assumption that the orbits are circular.

Solution: Since the system is eclipsing, we are viewing the orbits edge-on, and the measured velocity amplitudes are the speeds of the stars in their orbits. This leaves us with three unknowns, for which we need three independent equations:

1. The stars revolve once per orbit (per period), so

$$P = \frac{2\pi r}{v_1 + v_2}$$
$$\boxed{r = \frac{P}{2\pi}(v_1 + v_2)}$$

2. Momentum conservation applies, so

$$\frac{m_1}{m_2} = \frac{v_2}{v_1}$$
$$m_2 = m_1 \frac{v_1}{v_2}$$

3. Kepler's third law applies, so

$$\begin{aligned}
P^2 &= \frac{4\pi^2 r^3}{G(m_1 + m_2)} \\
m_1 + m_2 &= \frac{4\pi^2 r^3}{GP^2} \\
m_1\left(1 + \frac{v_1}{v_2}\right) &= \\
m_1 \frac{v_1 + v_2}{v_2} &= \\
m_1 &= \frac{4\pi^2 r^3}{GP^2}\frac{v_2}{v_1 + v_2} \\
&= \frac{4\pi^2}{GP^2}\left(\frac{P}{2\pi}(v_1 + v_2)\right)^3 \frac{v_2}{v_1 + v_2} \\
\Aboxed{m_1 &= \frac{Pv_2}{2\pi G}(v_1 + v_2)^2}
\end{aligned}$$

Plugging back in, we can solve for m_2:

$$m_2 = \frac{Pv_2}{2\pi G}(v_1 + v_2)^2 \frac{v_1}{v_2}$$
$$\boxed{m_2 = \frac{Pv_1}{2\pi G}(v_1 + v_2)^2}$$