Astronomy 142 — Practice Midterm Exam #1

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Name: ____

You may consult *only* one page of formulas and constants and a calculator while taking this test. You may *not* consult any books, digital resources, or each other. All of your work must be written on the attached pages, using the reverse sides if necessary. The final answers, and any formulas you use or derive, must be indicated clearly (answers must be circled or boxed). You will have one hour and fifteen minutes to complete the exam. Good luck!

- First, work on the problems you find the easiest. Come back later to the more difficult or less familiar material. Do not get stuck.
- The amount of space left for each problem is not necessarily an indication of the amount of writing it takes to solve it.
- Numerical answers are incomplete without units and should not be written with more significant figures than they deserve.
- Remember, you can earn partial credit for being on the right track. Be sure to show enough of your reasoning that we can figure out what you are thinking.

$R_\odot = 6.96 \times 10^{10}$ cm	$M_{\text{bol}} = 4.74$
$M_\odot = 1.989 \times 10^{33}$ g	$m_V = -26.71$
$L_\odot = 3.827 \times 10^{33}$ erg/s	$M_V = 4.86$
$T_e = 5772$ K	$BC_V = -0.12$
1 AU $= 149,597,870$ km	1 pc $= 206,265$ AU
$k = 1.38 \times 10^{-16}$ erg/K	$\sigma = 5.6704 \times 10^{-5}$ erg s^{-1} cm^{-2} K^{-4}
$G = 6.674 \times 10^{-8}$ dyn cm^2 g^{-2}	$c = 3 \times 10^{10}$ cm/s
$h = 6.6261 \times 10^{-27}$ erg s	$m_p = 1.6726 \times 10^{-24}$ g
$m_n = 1.6749 \times 10^{-24}$ g	$m_e = 9.1094 \times 10^{-28}$ g
$q_e = 4.803 \times 10^{-10}$ esu	

1. **Short answers**: Please write in complete sentences, and feel free to use equations and/or sketches to help explain your thoughts.

   (a) (5 points) A celestial object is above the horizon for exactly 12 hours. What is its declination?

   Solution: For an object above the horizon for exactly 12h, it will be at the horizon (altitude $= 0°$) when its HA $=$ 6h. The zenith angle is the complement of the altitude, so an altitude of 0 corresponds to a zenith angle

   $$\mathrm{ZA} = 90° - \text{altitude} = 90°$$

   Therefore,

   $$\begin{aligned} \cos \mathrm{ZA} &= \cos\delta \cos\lambda \cos \mathrm{HA} + \sin\delta \sin\lambda \\ \cos 90° &= \cos\delta \cos\lambda \cos 6^{\mathrm{h}} + \sin\delta \sin\lambda \\ 0 &= \sin\delta \sin\lambda \end{aligned}$$

   So either $\delta = 0°$ or $\lambda = 0°$. Since the question does not specify where we are, the more general answer is that $\boxed{\delta = 0°}$

   (b) (5 points) Explain why there is a maximum degeneracy pressure that electrons are capable of exerting, and therefore a maximum mass for white dwarf stars.

   Solution: This is directly traceable to the finite speed of light. To support a greater weight, the electrons in a degenerate gas need to each be confined to a smaller volume V — that is, the object needs to get smaller — so that, by the uncertainty principle, they will move faster and hit the walls of their "cells" harder. (In other, nonrelativistic, words, make $\Delta x \propto V^{1/3}$ smaller to make $\Delta p_x \geq \hbar/2\Delta x$ larger.) Since electrons cannot move faster than the speed of light, there is an eventual upper limit to the gained momentum uncertainty (and degeneracy pressure) from the reduction of the volume uncertainty.

(c) (5 points) Describe the difference in nature between the pulsations exhibited by the Sun — the "five minute oscillations" — and those found in Mira, Cepheid, and RR Lyrae variables.

Solution: The long-period variables (like Mira) and the variables lying along the instability strip in the H-R diagram exhibit purely radial oscillations dominated by the fundamental mode. Their boundary conditions involve a pressure maximum at the center and the first pressure node at the surface. The Solar five-minute oscillations are non-radial: the waves have components perpendicular to radius and thus rattle between the solar surface and a "turning point" in the interior whose depth depends upon how many pressure nodes are involved on the surface. Instead of having a wavelength determined by how many waves fit between center and surface, nonradial oscillations have wavelengths determined by how many waves fit in a complete circuit of the star.

(d) (5 points) Most of the double-line spectroscopic binaries that we see involve two stars of similar mass. Why?

Solution: We see the sum of the light from the two stars in a spectroscopic binary; we do not resolve them spatially. Luminosity increases very sharply with mass ($L \propto M^4$). Thus, if one member of the pair is less massive than the other one, it is *much* fainter, and its signal will be swamped by the signal of the more massive star; the signal will appear as a single-line spectroscopic binary.

2. Consider two fusion mechanisms: the main proton-proton chain and deuterium fusion. The masses of deuterium and helium are

$$m({}^{2}_{1}\mathrm{H}) = 3.3426 \times 10^{-24}\ \mathrm{g} \qquad\qquad m({}^{4}_{2}\mathrm{He}) = 6.6447 \times 10^{-24}\ \mathrm{g}$$

(a) (10 points) Fill in the blanks in the nuclear reaction chains for these two processes:

p-p chain I

$$2\,{}^{1}_{1}\mathrm{H} \rightarrow {}^{2}_{1}\mathrm{H} + \underline{\ {}^{\mathbf{0}}_{\mathbf{1}}\mathbf{e}^{+}\ } + \underline{\ {}^{\mathbf{0}}_{\mathbf{0}}\nu_{e}\ } \quad (2\times)$$

$${}^{2}_{1}\mathrm{H} + \underline{\ {}^{\mathbf{1}}_{\mathbf{1}}\mathbf{H}\ } \rightarrow {}^{3}_{2}\mathrm{He} + \underline{\ {}^{\mathbf{0}}_{\mathbf{0}}\gamma\ } \quad (2\times)$$

$$2\,{}^{3}_{2}\mathrm{He} \rightarrow \underline{\ {}^{\mathbf{4}}_{\mathbf{2}}\mathbf{He}\ } + 2\,{}^{1}_{1}\mathrm{H}$$

$$\text{Total: } \underline{\ \mathbf{4}\,{}^{\mathbf{1}}_{\mathbf{1}}\mathbf{H}\ } \rightarrow \underline{\ {}^{\mathbf{4}}_{\mathbf{2}}\mathbf{He}\ } + \underline{\ \mathbf{2}\,{}^{\mathbf{0}}_{\mathbf{1}}\mathbf{e}^{+}\ } + \underline{\ \mathbf{2}\,{}^{\mathbf{0}}_{\mathbf{0}}\nu_{e}\ } + \underline{\ \mathbf{2}\,{}^{\mathbf{0}}_{\mathbf{0}}\gamma\ }$$

d-d fusion :

$$2\,{}^{2}_{1}\mathrm{H} \rightarrow {}^{4}_{2}\mathrm{He} + \underline{\ {}^{\mathbf{0}}_{\mathbf{0}}\gamma\ }$$

(b) (5 points) Calculate the energy $\Delta E_{\text{p-p}}$ and $\Delta E_{\text{d-d}}$ released by fusion and available for the lightweight products (electrons, photons, etc.) in each of the "total" reactions. Express your answer in ergs.

Solution:

$$\Delta E_{p-p} = \left(m({}^{4}_{2}\mathrm{He}) - 4m({}^{1}_{1}\mathrm{H})\right) c^2 = 4.12 \times 10^{-5}\ \mathrm{erg}$$
$$\Delta E_{d-d} = \left(m({}^{4}_{2}\mathrm{He}) - 2m({}^{2}_{1}\mathrm{H})\right) c^2 = 3.82 \times 10^{-5}\ \mathrm{erg}$$

(c) (5 points) Compare the neutrino production in these two mechanisms. Which is a better explanation of the detected quantity of solar neutrinos: neutrino oscillation or the Sun being made of pure deuterium? In other words, if the core of the Sun were made of pure deuterium, would the original solar neutrino problem be solved or does neutrino oscillation still look like a better explanation?

Solution: Solar neutrinos are detected at about one-third the rate that the p–p chains produce them (see above; each ${}^{4}_{2}\mathrm{He}$ comes with two electron neutrinos in p–p I). This deficit is the celebrated solar-neutrino problem. Deuterium fusion, on the other hand, produces no neutrinos, so the observations would be a huge excess over expectations — an infinite excess, in ratio terms. Thus, pure deuterium, while it would certainly produce fewer neutrinos than p–p, would make the situation worse. Looks like p–p fusion and neutrino oscillations work better.

3. **A deuterium star**. A $1M_\odot$ main-sequence star like the Sun has a central temperature of 1.57×10^7 K and the subatomic particles in its center (electrons, protons, and other ions) have an average mass of $0.62m_p$. Consider a $1M_\odot$ star made completely of deuterium ${}^2_1$H, which produces energy by d-d fusion into helium. Assume its internal structure is like that of the Sun, apart from its different internal composition.

(a) (5 points) Calculate the average particle mass μ in the deuterium star's interior assuming that the material is fully ionized. Express your answer in terms of m_p.

Solution: The deuterium nucleus has a mass of about $2m_p$, since it contains a proton and neutron. Two particles — the nucleus and an electron — result from the ionization of deuterium. Thus,

$$\mu = \frac{m_d + m_e}{2} \approx \frac{m_d}{2} \approx m_p$$

(b) (10 points) Suppose the deuterium star has the same radius as the Sun. Calculate its central temperature.

Solution: Balance the central pressure with the ideal-gas pressure:

$$\begin{aligned} P_C &\propto \frac{GM^2}{R^4} = \frac{\rho k T_C}{\mu} \propto \frac{MkT_C}{R^3\mu} \\ T_C &\propto \frac{GM\mu}{kR} \end{aligned}$$

Write this again for the Sun and divide the two expressions:

$$\begin{aligned} \frac{T_C}{T_{C,\odot}} &= \frac{M\mu}{R}\frac{R_\odot}{M_\odot\mu_\odot} = \frac{\mu}{\mu_\odot} \\ T_C &= T_{C,\odot}\frac{\mu}{\mu_\odot} = 2.53 \times 10^7 \text{ K} \end{aligned}$$

(c) (10 points) Show that at this temperature, d-d fusion reactions in the core of the deuterium star occur a bit less than half as often as p-p reactions in the Sun's core.

Solution: The rate at which fusion reactions take place is proportional to the fusion probability.

$$p = Ae^{-(T_0/T)^{1/3}} \tag{1}$$

where

$$T_0 = \left(\frac{3}{2}\right)^3 \left(\frac{8\pi q_1 q_2}{h}\right)^2 \frac{m_{red}}{k} \tag{2}$$

and where the qs are the electric charges, and $m_{red} = m_1 m_2/(m_1 + m_2)$ is the reduced mass of the fusing nuclei. Now, $q_1 = q_2 = q_e$ for both p–p and d–d reactions, and the reduced masses are just half the mass of the fusing nuclei:

$$m_{red} = \frac{m_p m_p}{m_p + m_p} = \frac{m_p}{2} = \mu \tag{3}$$

for p–p, and similarly $m_{red} = m_d/2$ for d–d. Thus,

$$T_{0,d-d} = \left(\frac{3}{2}\right)^3 \left(\frac{8\pi q_e^2}{h}\right)^2 \frac{m_d}{2k} = 3.131 \times 10^{10} \text{ K}$$

$$T_{0,p-p} = \left(\frac{3}{2}\right)^3 \left(\frac{8\pi q_e^2}{h}\right)^2 \frac{m_p}{2k} = 1.565 \times 10^{10} \text{ K}$$

and

$$\frac{p_{d-d}}{p_{p-p}} = \exp\left(-\left(\frac{T_{0,d-d}}{T_C}\right)^{1/3}\right) \exp\left(-\left(\frac{T_{0,p-p}}{T_{C,\odot}}\right)^{1/3}\right)$$

$$= \exp\left(-\left(\frac{3.131 \times 10^{10} \text{ K}}{2.53 \times 10^7 \text{ K}}\right)^{1/3}\right) \exp\left(-\left(\frac{1.565 \times 10^{10} \text{ K}}{1.57 \times 10^7 \text{ K}}\right)^{1/3}\right)$$

$$\frac{p_{d-d}}{p_{p-p}} = 0.476$$

4. You have observed an eclipsing double-line spectroscopic binary in which the two sets of lines are identical, but for which the Doppler shifts are sinusoidal and opposite. The velocity amplitudes are both $v = 43$ km/s and the period $P = 31$ days. The visual magnitude of the binary is $V = 10$.

   (a) (10 points) Make some reasonable assumptions and deductions about the shape and orientation of the orbits and the relative masses of the stars in the binary system. Then calculate the separation of the stars. Express your answer in terms of $R_\odot$.

   Solution: First, note that the orbits are viewed edge on (eclipsing) and are circular (sinusoidal Doppler shifts), and that the stars have equal effective temperature (identical spectra). If they are both in the same stage of development, this means that their masses are equal.

   You can either use the formula that we derived in class or, more simply, note that each star travels in the same circle, for which the circumference is vP and the radius is $vP/2\pi$. The stars are always on opposite sides of the circle (center of mass at the center of the circle!), so they are separated by the circle's diameter a:

   $$a = 2r = \frac{vP}{\pi} = 3.67 \times 10^{12} \text{ cm} = 52.7 R_\odot \tag{4}$$

   (b) (5 points) Calculate the masses of the stars in the binary system, expressing your answer in terms of $M_\odot$.

   Solution: Use Kepler's third law:

   $$\begin{aligned} M_1 + M_2 = 2M &= \frac{4\pi^2}{GP^2} a^3 \\ M &= \frac{2\pi^2}{GP^2} a^3 = 1.02 M_\odot \qquad \text{(each star)} \end{aligned}$$

(c) (10 points) Calculate the visual magnitude of each star.

Solution: The stars have the same magnitude V and the same flux f. Fluxes add, but magnitudes do not. If the flux of a zero-magnitude star is f_0, then applying $m_1 - m_2 = 2.5\log(f_2/f_1)$ to the binary and the zero-magnitude star gives

$$10 = 2.5\log\left(\frac{f_0}{2f}\right)$$
$$\frac{f_0}{f} = 2 \times 10^4$$

Applying $m_1 - m_2 = 2.5\log(f_2/f_1)$ again to this newly-found flux ratio gives

$$V = 2.5\log\left(\frac{f_0}{f}\right) = 2.5\log\left(2 \times 10^4\right) = 10.75 \qquad (5)$$

5. A $1M_\odot$ star is observed to oscillate in brightness with a period of 48 hr and an amplitude of 1 visual magnitude.

   (a) (10 points) Assume the star is uniform in density. What is its radius in terms of $R_\odot$?

   Solution:

   $$\Pi = \sqrt{\frac{6\pi}{\gamma G\rho}} = \sqrt{\frac{6\pi}{\gamma G}\frac{4\pi R^3}{3M}}$$

   $$R = \left(\frac{\gamma G M \Pi^2}{8\pi^2}\right)^{1/3} = \left(\frac{(5)(6.674 \times 10^{-8}\ \text{dyn cm}^2\ \text{g}^{-2})(1.989 \times 10^{33}\ \text{g})(48 \times 60 \times 60\ \text{s})^2}{(3)(8\pi^2)}\right)^{1/3}$$

   $$R = 4.37 \times 10^{11}\ \text{cm} = 6.28R_\odot$$

   (b) (5 points) In what phase of its evolution is the star likely to be?

   Solution: Large enough to be well off the main sequence, but not huge; 2-day period and 1-mag amplitude; all this sounds like an RR Lyrae star, so it must be on the horizontal branch (i.e. the helium-burning main sequence).