Astronomy 142 — Practice Final Exam

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You may consult two pages of formulas and constants and a calculator while taking this test. You may *not* consult any books, digital resources, or each other. All of your work must be written on the attached pages, using the reverse sides if necessary. The final answers, and any formulas you use or derive, must be indicated clearly (answers must be circled or boxed). You will have three hours to complete the exam. Good luck!

- First, work on the problems you find the easiest. Come back later to the more difficult or less familiar material. Do not get stuck.
- The amount of space left for each problem is not necessarily an indication of the amount of writing it takes to solve it.
- Numerical answers are incomplete without units and should not be written with more significant figures than they deserve.
- You must show your work and/or explain your answers to receive full credit.
- Remember, you can earn partial credit for being on the right track. Be sure to show enough of your reasoning that we can figure out what you are thinking.

$R_{\odot} = 6.96 \times 10^{10} \text{ cm}$	$M_{ m bol} = 4.74$
$M_{\odot} = 1.989 \times 10^{33} \text{ g}$	$m_V = -26.71$
$L_{\odot} = 3.827 \times 10^{33} \text{ erg/s}$	$M_V = 4.86$
$T_e = 5772 \text{ K}$	$BC_V = -0.12$
$R_{\oplus} = 6.371 \times 10^8 \text{ cm}$	$M_{\oplus} = 5.9736 \times 10^{27} \ {\rm g}$
1 AU = 149,597,870 km	1 pc = 206, 265 AU
$k = 1.38 \times 10^{-16} \text{ erg/K}$	$\sigma = 5.6704 \times 10^{-5} \ {\rm erg \ s^{-1} \ cm^{-2} \ K^{-4}}$
$G = 6.674 \times 10^{-8} \text{ dyn cm}^2 \text{ g}^{-2}$	$c = 3 \times 10^{10} \text{ cm/s}$
$H_0 = 73.04 \text{ km/s/Mpc}$	$h = 6.6261 \times 10^{-27} \text{ erg s}$
$m_p = 1.6726 \times 10^{-24} \text{ g} = 938.3 \text{ MeV}/c^2$	$m_n = 1.6749 \times 10^{-24} \text{ g} = 939.6 \text{ MeV}/c^2$
$m_e = 9.1094 \times 10^{-28} \text{ g} = 0.511 \text{ MeV}/c^2$	$e = 4.803 \times 10^{-10} \text{ cm}^{3/2} \text{ g}^{1/2} \text{ s}^{-1}$

- 1. Answer each of the following with a few complete sentences. Feel free to use diagrams and/or equations if you feel that they will help you answer the questions.
 - (a) (5 points) An object is observed at redshift z. What is the dimensionless scale factor which applies to this object?

Solution:	
	$a = \frac{1}{1+z}$
	1 2

(b) (5 points) How old is the Universe, according to current best estimates?

Solution: The age is 13.8 Gyr according to CMB data and our best guess model of the Universe, using the SH0ES value of $H_0 = 73.04$ km s⁻¹ Mpc⁻¹.

(c) (5 points) Under what conditions can the virial theorem be used to measure the mass of a cluster of stars or galaxies?

Solution: The system must be in dynamic (thermal) equilibrium for the virial theorem to be applied. A system is considered to be in equilibrium when its age is older than its relaxation time.

(d) (5 points) Who invented the use of standard candles, and how was it done?

Solution: Henrietta Leavitt, one of the "computers" employed by Edward Pickering at Harvard College Observatory. She discovered the relation between the apparent magnitude and pulsation period of Cepheids in the Magellanic clouds and realized that since all the Cepheids in each cloud are at the same distance from us, this is really a period-luminosity relation.

(e) (5 points) Calculate the average density in g/cm^3 of a classical Cepheid variable star with a 10-day period.



(f) (5 points) With what speed do galaxies 100 Mpc away recede from us? What is their redshift?

Solution:

 $v = H_0 d = (73.04 \text{ km/s/Mpc})(100 \text{ Mpc})$ v = 7304 km/s

$$z \approx \frac{v}{c} = \frac{7304 \text{ km/s}}{3 \times 10^5 \text{ km/s}}$$
$$z \approx 0.024$$

2. The distance ladder

(a) (10 points) In the outer reaches of a certain galaxy, a variable star is detected with a period of 30 days and a total flux of 3.3×10^{-15} erg s⁻¹ cm⁻². Under the assumption that the star is a Cepheid I, calculate the distance to the galaxy.

Solution: According to Leavitt's Law,

$$\overline{M_V} = -2.77 \log \Pi - 1.69$$

Since the Cepheid I stars are located on the Instability Strip of the H-R diagram, they are comprised of mainly F and G stars which have zero bolometric correction. Therefore, $\overline{M_{\text{bol}}} = \overline{M_V}$, so

$$\overline{M_{\text{bol}}} = -2.77 \log \Pi - 1.69$$

$$4.75 - 2.5 \log \left(\frac{L}{L_{\odot}}\right) =$$

$$4.75 - 2.5 \log \left(\frac{4\pi r^2 \bar{f}}{L_{\odot}}\right) =$$

$$r = \sqrt{\frac{L_{\odot}}{4\pi \bar{f}}} 10^{(1.108 \log \Pi + 2.576)}} = \sqrt{\frac{3.827 \times 10^{33} \text{ erg/s}}{4\pi (3.3 \times 10^{-15} \text{ erg/s/cm}^2)}} 10^{(1.108 \log(30) + 2.576)}}$$

$$r = 3.88 \times 10^{25} \text{ cm} = 12.6 \text{ Mpc}$$

(b) (10 points) In the same galaxy, a supernova remnant is monitored for two years. Visible spectral lines show that during this time the gas has a maximum velocity along the line of sight of 5000 km/s. Radio images show the remnant to be circular in appearance. During the two-year period, the diameter of the remnant increases by 0.84×10^{-3} arcseconds. Using these data, calculate the distance to the galaxy.

Solution: The increase in radius over two years was

$$\Delta R = vt = (5000 \text{ km/s})(2 \text{ yr})(3.16 \times 10^7 \text{ s/yr})$$
$$\Delta R = 3.16 \times 10^{11} \text{km} = 3.16 \times 10^{16} \text{ cm}$$

During this same time, the increase in angular diameter observed was

$$\Delta \theta = \frac{2\Delta R}{r}$$

$$r = \frac{2\Delta R}{\Delta \theta} = \frac{2(3.16 \times 10^{16} \text{ cm})}{(0.84 \times 10^{-3} \text{ arcsec})(648000/\pi \text{ rad/arcsec})}$$

$$r = 1.5 \times 10^{25} \text{ cm} = 5 \text{ Mpc}$$

(c) (5 points) Which of these measurements is likely to be more accurate? If the two estimates are different, describe the source of the discrepancy.

Solution: The method used in part b is geometric and therefore a much more reliable distance measurement than the Cepheid-based method, which relies upon correct identification of the star, accurate measurement of its magnitudes, and calibration of Leavitt's Law. Thus, the 5 Mpc determination is more likely to be correct.

- 3. Consider a universe in which $\Lambda = 0$ and the normalized matter density is $\Omega = \Omega_{M_0} > 1$.
 - (a) (20 points) Derive an expression for the comoving radial distance between z = 0 (that is, today) and $z_d = 1090$, the redshift of the decoupling surface.

Solution: The comoving radial distance is that traveled by light between decoupling and us, which by the Robertson-Walker interval is given by

$$0 = ds^{2} = c^{2}dt^{2} - a^{2}dr^{2}$$
$$a^{2}dr^{2} = c^{2}dt^{2}$$
$$\int_{r_{d}}^{r} dr' = c\int_{t_{d}}^{t} \frac{dt'}{a}$$
$$\Delta r = c\int_{t_{d}}^{t} \frac{dt'}{a}\frac{da}{da}$$
$$\Delta r = c\int_{a_{d}}^{a} \frac{da}{a\dot{a}}$$

We can use the Friedmann equation to replace \dot{a} .

$$\begin{split} \Delta r &= \frac{c}{H_0} \int_{a_d}^1 \frac{da}{a\sqrt{1+\Omega\left(\frac{1}{a}-1\right)}} \\ &= \frac{c}{H_0\sqrt{\Omega}} \int_{a_d}^1 \frac{da}{\sqrt{a\left(1-a\frac{\Omega-1}{\Omega}\right)}} \\ &= \frac{c}{H_0\sqrt{\Omega}} \sqrt{\frac{\Omega}{\Omega-1}} \int_{(\Omega-1)a_d/\Omega}^{(\Omega-1)/\Omega} \frac{dx}{\sqrt{x(1-x)}} \\ &= \frac{2c}{H_0\sqrt{\Omega-1}} \int_{\sin^{-1}\sqrt{(\Omega-1)a_d/\Omega}}^{\sin^{-1}\sqrt{(\Omega-1)a_d/\Omega}} \frac{\cos\theta\sin\theta}{\sqrt{\sin^2\theta(1-\sin^2\theta)}} d\theta \\ &= \frac{2c}{H_0\sqrt{\Omega-1}} \int_{\sin^{-1}\sqrt{(\Omega-1)a_d/\Omega}}^{\sin^{-1}\sqrt{(\Omega-1)a_d/\Omega}} d\theta \\ &= \frac{2c}{H_0\sqrt{\Omega-1}} \left[\sin^{-1}\sqrt{\frac{\Omega-1}{\Omega}} - \sin^{-1}\sqrt{a_d\frac{\Omega-1}{\Omega}} \right] \\ \Delta r &= \frac{2c}{H_0\sqrt{\Omega-1}} \left[\sin^{-1}\sqrt{\frac{\Omega-1}{\Omega}} - \sin^{-1}\sqrt{\frac{1+z_d}{\Omega}} \frac{\Omega-1}{\Omega} \right] \end{split}$$

(b) (5 points) Independent of the curvature of the universe, the acoustic horizon size is given as

$$\ell_{d} = \frac{2c}{3H_{0}}\sqrt{\frac{a_{d}}{\Omega_{M_{0}}}} = \frac{2c}{3H_{0}\sqrt{\Omega_{M_{0}}(1+z_{d})}}$$

Calculate the angular size, in degrees, of the fundamental mode of acoustic oscillations visible on the decoupling surface as the cosmic-background anisotropies, using the results of part a and a normalized matter density of $\Omega_{M_0} = 2$. How does your answer compare to the measured angular size of the fundamental mode, $\theta_d = 0.66^{\circ}$?

Solution:

$$\begin{aligned} \theta_d &= \frac{\ell_d}{\Delta r} \\ &= \frac{2c}{3H_0\sqrt{\Omega(1+z_d)}} \frac{H_0\sqrt{\Omega-1}}{2c} \left[\sin^{-1}\sqrt{\frac{\Omega-1}{\Omega}} - \sin^{-1}\sqrt{\frac{1}{1+z_d}} \frac{\Omega-1}{\Omega} \right]^{-1} \\ &= \frac{1}{3}\sqrt{\frac{\Omega-1}{\Omega} \frac{1}{1+z_d}} \left[\sin^{-1}\sqrt{\frac{\Omega-1}{\Omega}} - \sin^{-1}\sqrt{\frac{1}{1+z_d}} \frac{\Omega-1}{\Omega} \right]^{-1} \\ &= \frac{1}{3}\sqrt{\frac{2-1}{2} \frac{1}{1+1090}} \left[\sin^{-1}\sqrt{\frac{2-1}{2}} - \sin^{-1}\sqrt{\frac{1}{1+1090}} \frac{2-1}{2} \right]^{-1} \\ \theta_d &= 9.3 \times 10^{-3} \text{ rad} = 0.54^\circ \end{aligned}$$

This is smaller than the measured value of 0.6° .

- 4. A $1M_{\odot}$ star with radius $1R_{\odot}$ has just reached the end of its main-sequence life, with a central temperature of 1.4×10^7 K and a core containing 10% of its total mass. Hydrogen fusion has recently shut off in the core, which we assume for simplicity to be composed completely of ${}_{2}^{4}$ He.
 - (a) (5 points) What is the central pressure (in $dyne/cm^2$) due to the weight of the outer layers of the star?

Solution: For stars with low to moderate mass, the central pressure is approximately
$$\begin{split} P_C &\approx 19 \frac{GM^2}{R^4} = 19 \frac{(6.674 \times 10^{-8} \text{ dyn cm}^2 \text{ g}^{-2})(1.989 \times 10^{33} \text{ g})^2}{(6.96 \times 10^{10} \text{ cm})^4} \\ P_C &\approx 2.14 \times 10^{17} \text{ dyn/cm}^2 \end{split}$$

(b) (5 points) At the very center of the star, nonrelativistic electron degeneracy pressure balances gravity. What is the density of electrons there?

Solution: Electron degeneracy pressure is equal to $P = 0.0485 \frac{h^2 n_e^{5/3}}{-}$

$$n_e = \left(\frac{m_e P_C}{0.0485h^2}\right)^{3/5} = \left(\frac{(9.109 \times 10^{-28} \text{ g})(2.14 \times 10^{17} \text{ dyn/cm}^2)}{0.0485(6.6261 \times 10^{-27} \text{ erg s})^2}\right)^{3/5}$$
$$n_e = 2.38 \times 10^{26} \text{ cm}^{-3}$$

(c) (5 points) Suppose hydrogen fusion starts up in the shell just outside the helium core. The shell is the same temperature as the core itself. As this fusion proceeds, helium "ash" is added to the core. The temperature scales with the core mass M_C according to $T_C \propto M_C^{4/3}$. What is the mass of the helium core when the temperature reaches 10^8 K, high enough to ignite ⁴/₂He fusion?

Solution:

$$T_C \propto M_C^{4/3}$$

$$\frac{T_C}{T_{C,\text{initial}}} = \left(\frac{M_C}{M_{C,\text{initial}}}\right)^{4/3}$$

$$M_C = M_{C,\text{initial}} \left(\frac{T_C}{T_{C,\text{initial}}}\right)^{3/4} = 0.1 M_{\odot} \left(\frac{10^8 \text{ K}}{1.4 \times 10^7 \text{ K}}\right)^{3/4}$$

$$M_C = 0.44 M_{\odot}$$

- 5. There are three different basic types of Big Bang universes.
 - (a) (15 points) What are their pasts and futures, if they are matter-dominated?

Solution: The three different types of universes are an open, flat, and closed universe. All three universes start out expanding. In an open universe, the expansion velocity is greater than that of its "escape velocity," so its kinetic energy is greater than its potential energy. As a result, it will continue to accelerate as its expands. From today's viewpoint, this universe would be the oldest.

The flat universe has an expansion velocity equal to that of the "escape velocity," so the total kinetic energy is equal to the total potential energy. This universe will constantly expand, but it will continue to do so at a constant rate.

The closed universe is the most short-lived of them all. Having an expansion velocity less than the "escape velocity," its kinetic energy is less than the potential energy. This results in a negative acceleration rate; at some point, this universe will stop expanding and recompress back to its original state. (b) (5 points) What determines the fate of a universe?

Solution: The relationship between the expansion velocity and the "escape velocity" in each universe determines the universe's fate.

(c) (5 points) What are the current ideas about the fate of our Universe?

Solution: Our universe is thought to be a flat universe that is dominated by dark energy. As a result, its expansion rate will increase over time, eventually separating everything by an infinite amount.

6. Rotation curves

(a) (10 points) Draw a typical rotation curve for a spiral galaxy; label the axes and identify critical values of the curve on the axes. Indicate the different types of rotation observed.



(b) (5 points) Describe a rotation curve. What information can be gleaned from it?

Solution: A rotation curve of a spiral galaxy is a graphical representation of the velocity of the elements of a galaxy as a function of the radius.

Rotation curves can help us to estimate the mass of the galaxy, since we know both the radius and speed at a given point. They also help to prove the existence of dark matter, since the classical prediction of the rotation curve for a spiral galaxy would be $V \propto R^{-2}$. The difference between the two curves can only be explained by dark matter.

7. Galaxy surface brightness

(a) (5 points) A spiral galaxy appears circular when seen face-on. The flux observed per steradian of the galaxy is given by the relation

$$\Sigma(r) = \Sigma_0 e^{-r/r_0}$$

where r is the angular distance in radians from the center of the galaxy. What is the total flux that we observe from the galaxy?

Solution: The total flux is obtained by integrating the surface brightness profile over all radii, $F = \int_0^\infty \Sigma(r) 2\pi r \ dr$ $= 2\pi \Sigma_0 \int_0^\infty e^{-r/r_0} r \ dr$ $\overline{F = 2\pi \Sigma_0 r_0^2}$ (b) (10 points) You observe an elliptical E0 galaxy. The flux you measure per steradian is given by the relation

$$\Sigma(r) = \Sigma_0 \exp\left[-(r/r_0)^{1/4}\right]$$

where r is the angular distance in radians from the center of the galaxy. What is the total observed flux? *Hint: You may wish to use the fact that*

$$\int_0^\infty x^n e^{-x} dx = n!$$

Solution: The total flux is obtained by integrating the surface brightness profile over all radii. However, the radial dependence is a bit more complex so we need to simplify the integral

$$F = 2\pi\Sigma_0 \int_0^\infty e^{-(r/r_0)^{1/4}} r \, dr$$

using the substitution

$$u = (r/r_0)^{1/4}$$
 $r = r_0 u^4$ $dr = 4r_0 u^3 du$

This gives

$$F = 2\pi\Sigma_0 \int_0^\infty e^{-(r/r_0)^{1/4}} r \, dr$$

= $2\pi\Sigma_0 \int_0^\infty e^{-u} r_0 u^4 \, 4r_0 u^3 \, du$
= $8\pi\Sigma_0 r_0^2 \int_0^\infty u^7 e^{-u} \, du$
= $8\pi\Sigma_0 r_0^2 \cdot 7!$
 $F = 8!\pi\Sigma_0 r_0^2$

8. The mass density of a star with mass M and radius R decreases linearly from the center to the surface of the star and vanishes at the surface:

$$\rho(r) = \rho_C \left(1 - \frac{r}{R} \right)$$

(a) (10 points) Derive an expression for the density ρ_C at the center of the star, in terms of M and R.

Solution:

$$M = \int_0^R \rho(r) 4\pi r^2 dr$$

= $4\pi\rho_C \int_0^R r^2 dr - \frac{4\pi\rho_C}{R} \int_0^R r^3 dr$
= $\frac{4\pi\rho_C}{3} R^3 - \frac{4\pi\rho_C}{4R} R^4$
= $\frac{\pi\rho_C R^3}{3}$
 $\rho_C = \frac{3M}{\pi R^3}$

(b) (20 points) Derive an expression for the pressure P_C at the center of the star, in terms of M and R.

Solution: Start with hydrostatic equilibrium:

$$P(R) - P_C = \int_0^R \frac{dP}{dr} dr = -\int_0^R \frac{GM(r)\rho(r)}{r^2} dr$$
$$P_C = \int_0^R \frac{GM(r)\rho(r)}{r^2} dr$$

Now we need an expression for the mass M(r) contained within the radius r, which we get by integrating the density between 0 and r:

$$M(r) = \int_{0}^{r} \rho(r') 4\pi r'^{2} dr'$$

= $\frac{12M}{R^{3}} \int_{0}^{r} \left(1 - \frac{r'}{R}\right) r'^{2} dr'$
= $\frac{12M}{R^{3}} \left[\frac{r'^{3}}{3} - \frac{r'^{4}}{4R}\right]_{0}^{r}$
 $M(r) = 4M \left(\frac{r}{R}\right)^{3} \left[1 - \frac{3}{4}\left(\frac{r}{R}\right)\right]$

Now plug this into the hydrostatic equilibrium integral above:

$$P_{C} = \int_{0}^{R} \frac{GM(r)\rho(r)}{r^{2}} dr$$

$$= \frac{4GM}{R^{3}} \frac{3M}{\pi R^{3}} \int_{0}^{R} r\left(1 - \frac{3}{4}\frac{r}{R}\right) \left(1 - \frac{r}{R}\right) dr$$

$$= \frac{12GM}{\pi R^{6}} \int_{0}^{R} \left(r - \frac{7}{4}\frac{r^{2}}{R} + \frac{3}{4}\frac{r^{3}}{R^{2}}\right) dr$$

$$= \frac{12GM^{2}}{\pi R^{6}} \left[\frac{1}{2}r^{2} - \frac{7}{12}\frac{r^{3}}{R} + \frac{3}{16}\frac{r^{4}}{R^{2}}\right]_{0}^{R}$$

$$P_{C} = \frac{5GM^{2}}{4\pi R^{4}}$$