Astronomy 142 — Practice Final Exam

Professor Kelly Douglass

Spring 2024

Name: ____

You may consult two pages of formulas and constants and a calculator while taking this test. You may *not* consult any books, digital resources, or each other. All of your work must be written on the attached pages, using the reverse sides if necessary. The final answers, and any formulas you use or derive, must be indicated clearly (answers must be circled or boxed). You will have three hours to complete the exam. Good luck!

- First, work on the problems you find the easiest. Come back later to the more difficult or less familiar material. Do not get stuck.
- The amount of space left for each problem is not necessarily an indication of the amount of writing it takes to solve it.
- Numerical answers are incomplete without units and should not be written with more significant figures than they deserve.
- You must show your work and/or explain your answers to receive full credit.
- Remember, you can earn partial credit for being on the right track. Be sure to show enough of your reasoning that we can figure out what you are thinking.

$$R_\odot = 6.96 \times 10^{10} \text{ cm} \qquad M_{\rm bol} = 4.74$$
$$M_\odot = 1.989 \times 10^{33} \text{ g} \qquad m_V = -26.71$$
$$L_\odot = 3.827 \times 10^{33} \text{ erg/s} \qquad M_V = 4.86$$
$$T_e = 5772 \text{ K} \qquad BC_V = -0.12$$
$$R_\oplus = 6.371 \times 10^{8} \text{ cm} \qquad M_\oplus = 5.9736 \times 10^{27} \text{ g}$$
$$1 \text{ AU} = 149,597,870 \text{ km} \qquad 1 \text{ pc} = 206,265 \text{ AU}$$
$$k = 1.38 \times 10^{-16} \text{ erg/K} \qquad \sigma = 5.6704 \times 10^{-5} \text{ erg s}^{-1} \text{ cm}^{-2} \text{ K}^{-4}$$
$$G = 6.674 \times 10^{-8} \text{ dyn cm}^2 \text{ g}^{-2} \qquad c = 3 \times 10^{10} \text{ cm/s}$$
$$H_0 = 73.04 \text{ km/s/Mpc} \qquad h = 6.6261 \times 10^{-27} \text{ erg s}$$
$$m_p = 1.6726 \times 10^{-24} \text{ g} = 938.3 \text{ MeV}/c^2 \qquad m_n = 1.6749 \times 10^{-24} \text{ g} = 939.6 \text{ MeV}/c^2$$
$$m_e = 9.1094 \times 10^{-28} \text{ g} = 0.511 \text{ MeV}/c^2 \qquad e = 4.803 \times 10^{-10} \text{ cm}^{3/2} \text{ g}^{1/2} \text{ s}^{-1}$$

1. Answer each of the following questions with a few complete sentences. Feel free to use diagrams and/or equations if you feel that they will help you answer the questions.

   (a) (5 points) Why is the time told on a sundial different by several minutes from time told on a clock?

   Solution: The Earth does not move at a uniform speed through its orbit, and the plane of the Solar System does not coincide with the equator. For both of these reasons, the Sun does not move through the background of fixed stars at a constant rate, and its shadow will not move at the same pace as a clock.

   (b) (5 points) How is it possible for knots within quasar jets to appear to move faster than light?

   Solution: Because they are moving close to the speed of light in a direction close to our line of sight. If we take two pictures of the knots, the light that comprised the second picture had a significant head start, as it has moved substantially closer to us in the meantime. When we estimate speed by distance divided by time without accounting for this head start, we therefore overestimate the speed, and the overestimate can exceed the speed of light.

(c) (5 points) Why is the cosmic background radiation essentially isotropic?

Solution: Because it arises so much closer to the original Universal singularity than we are. All paths light can take from that early in the Universe can be thought to originate in the singularity and, as that was all there was to the Universe then, the neighborhood of the singularity, and the cosmic microwave background, spread isotropically across the sky.

(d) (5 points) Give two strong pieces of observational evidence for the presence of dark energy in the Universe.

Solution:

1. The Universe is flat between here/now and the epoch of decoupling, despite the fact that the normalized mass density of the Universe (luminous plus dark matter) is much less than the density of a flat matter-dominated universe.
2. Distant Type Ia supernovae are fainter (more distant) than expected for their redshift and the Hubble relation. Therefore, the expansion of the Universe appears to be accelerating.

(e) (5 points) An isolated black hole has a mass of 10^8 g. At what temperature does it radiate?

Solution:

$$T = \frac{hc^3}{16\pi^2 kGM} = \frac{(6.6261 \times 10^{-27}\ \text{erg s})(3 \times 10^{10}\ \text{cm/s})^3}{16\pi^2(1.38 \times 10^{-16}\ \text{erg/K})(6.674 \times 10^{-8}\ \text{dyn cm}^2\ \text{g}^{-2})(10^8\ \text{g})}$$

$$T = 1.23 \times 10^{18}\ \text{K}$$

(f) (5 points) Under what conditions can the Robertson-Walker absolute interval be used to calculate the properties of spacetime in a model universe?

Solution: The universe must be isotropic and homogeneous, and there must be no force other than gravity acting on the contents.

2. (a) (15 points) Suppose that the Universe were flat — as observed — but besides a very small amount of normal matter, it contains only dark energy. Starting from the Friedmann equation, derive an expression for the relation between time, t, and the normalized scale factor, a.

Solution: We are told that $k = 0$ and that $\Omega_{M0} \ll \Omega_{\Lambda 0}$ (and implicitly, that $\Lambda > 0$, as $\Omega_{\Lambda 0} \simeq 1$). With these, we have

$$\dot{a}^2 = H_0^2 \left[1 + (a^2 - 1)\right]$$
$$\left(\frac{da}{dt}\right)^2 = H_0^2 a^2$$

Integrating this from today, $t = t_0$ and $a = a_0 = 1$, to some time t,

$$\int_{t_0}^{t} dt' = \frac{1}{H_0} \int_{1}^{a(t)} \frac{da'}{a'}$$
$$t - t_0 = \frac{1}{H_0} \ln a(t)$$
$$\boxed{a(t) = e^{H_0(t - t_0)}}$$

(b) (15 points) Repeat for a flat, matter-dominated universe: one in which there is negligible dark energy but lots of matter.

Solution: Now $\Omega_{M0} \simeq 1$ and $\Omega_{\Lambda 0} \simeq 0$:

$$\begin{aligned}
\dot{a}^2 &= H_0^2 \left[1 + \left(\frac{1}{a} - 1\right)\right] \\
\left(\frac{da}{dt}\right)^2 &= \frac{H_0^2}{a} \\
\int_{t_0}^{t} dt' &= \frac{1}{H_0} \int_1^{a(t)} \sqrt{a'}\, da' \\
t - t_0 &= \frac{2}{3H_0}\left(a(t)^{3/2} - 1\right) \\
&\boxed{a(t) = \left(\frac{3H_0}{2}(t - t_0) + 1\right)^{2/3}}
\end{aligned}$$

(c) (5 points) Which of these universes expands faster? What is the fate of each of these universes?

Solution: The flat dark-energy-dominated universe expands much faster than the matter-dominated one, because an exponential grows much faster than the 2/3 power law.

Both of these universes will continue to expand forever, though.

3. Consider a cylindrical, rotating cluster of stars that is gravitationally bound. Both its radius R and thickness $H \ll R$ are much larger than the typical distance between stars, and the mass density ρ of the cluster is uniform on scales much larger than the typical distance between stars.

   (a) (5 points) What is the rotation speed V of the stars in the cluster as a function of the cylindrical radius r? Your answer should include only given quantities and known constants.

Solution: With ρ independent of r and H small, we have

$$\begin{aligned} \frac{GM(r)}{r^2} &= \frac{V^2}{r} \\ \frac{G}{r^2}\rho\pi r^2 H &= \\ &\boxed{V(r) = \sqrt{\pi G \rho r H}} \end{aligned}$$

(b) (10 points) What is the typical random stellar speed, v?

Solution: The thickness is determined by the balance between weight and the pressure corresponding to the vertical component of random motions. For a small piece of the cluster, with area A and full height H, the balance is characterized by

$$\begin{aligned} F &= \rho \overline{v_z^2} A \\ ma &= \\ \rho A H g_z &= \\ \rho A H (2\pi G \mu) &= \\ \overline{v_z^2} &= 2\pi G H \mu \end{aligned}$$

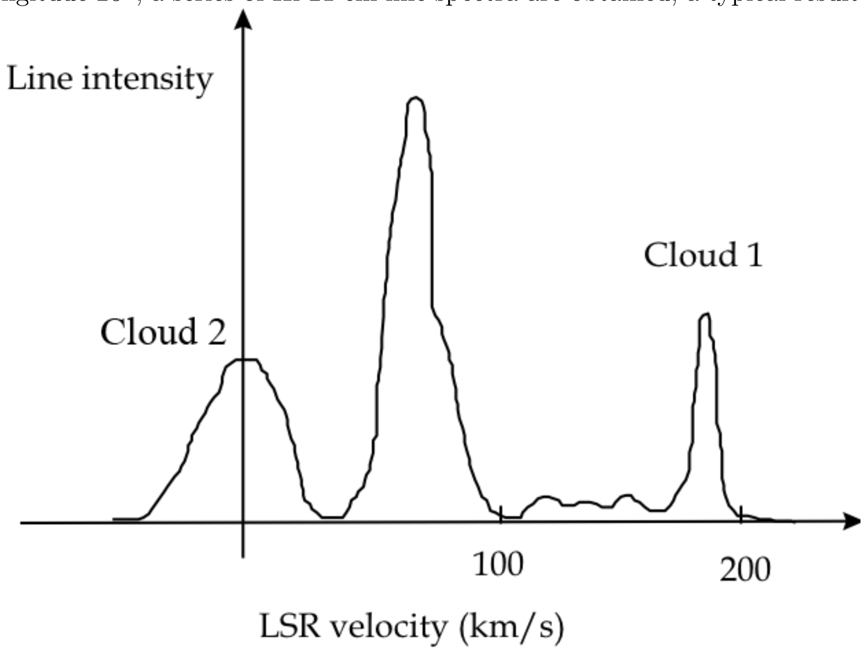
where μ is the mass per disk area, $\mu = \frac{\rho A H}{A} = \rho H$:

$$\overline{v_z^2} = 2\pi G \rho H^2$$

Thus,

$$\begin{aligned} \overline{v^2} &= \frac{\overline{v_z^2}}{\cos^2\theta} = 3\overline{v_z^2} = 6\pi G \rho H^2 \\ &\boxed{v = \sqrt{6\pi G \rho H^2}} \end{aligned}$$

4. At galactic longitude 20°, a series of HI 21 cm line spectra are obtained; a typical result is shown below.



HI line intensity as a function of radial velocity with respect to the local standard of rest.

(a) (10 points) The highest Doppler velocities correspond to a peak at 180 km/s and belong to a diffuse cloud we will refer to as Cloud 1 (see above figure). Calculate the distance r from Cloud 1 to the Galactic center, and the distance d from the Solar System to Cloud 1. Express your answers in kiloparsecs.

Solution: Cloud 1 has the highest redshift along the line of sight, which therefore must be tangent to this cloud's orbit. The Sun's orbit has a radius of 8.4 kpc, so

$$r = r_\odot \sin \ell = (8.4 \text{ kpc}) \sin 20^\circ = 2.9 \text{ kpc}$$
$$d = r_\odot \cos \ell = 7.9 \text{ kpc}$$

(b) (10 points) Calculate the Galactic rotation velocity of Cloud 1 and the Galactic mass enclosed by its orbit. *(Hint: $\Omega_\odot = 30$ km/s/kpc)*

Solution:

$$\begin{aligned} v &= v_{r,\max} + r_\odot \Omega(r_\odot) \sin \ell \\ &= 180 \text{ km/s} + (8.4 \text{ kpc})(30 \text{ km/s/kpc}) \sin 20^\circ \end{aligned}$$

$$\boxed{v = 266 \text{ km/s}}$$

$$M(r) = \frac{rv^2}{G} = \frac{(2.9 \times 10^3 \text{ pc})(3.09 \times 10^{18} \text{ cm/pc})(266 \times 10^5 \text{ cm/s})^2}{(6.674 \times 10^{-8} \text{ dyn cm}^2 \text{ g}^{-2})}$$

$$\boxed{M(r) = 9.6 \times 10^{43} \text{ g} = 4.8 \times 10^{10} M_\odot}$$

(c) (5 points) What is the galactocentric radius of Cloud 2?

Solution: Since it has zero velocity relative to us, it has the same galactocentric radius we do, 8.4 kpc:

$$\begin{aligned} v_r &= r_\odot[\Omega(r) - \Omega(r_\odot)] \sin \ell = 0 \\ \Omega(r) &= \Omega(r_\odot) \end{aligned}$$

$$\boxed{r = r_\odot}$$

5. A spherical clump of molecular hydrogen ($m_{H_2} = 3.3 \times 10^{-24}$ g) with mass $10M_\odot$ is observed to have a temperature $T = 10$ K and to be $r = 0.1$ pc in radius.

   (a) (5 points) Show that the clump is hydrostatically unstable and will collapse. Clearly state any assumptions that you make.

   Solution: You know of several ways to show this, but perhaps the quickest is to note that its mass is larger than the Jeans mass for its temperature, density, and composition. We can do no better than to assume that it is uniform in density.

$$\rho \approx \frac{3M}{4\pi r^3} = 1.6 \times 10^{-19} \text{ g/cm}^3$$
$$M_J = \left(\frac{kT}{m_{H_2} G}\right)^{3/2} \left(\frac{3}{4\pi\rho}\right)^{1/2} = 5.9 \times 10^{32} \text{ g} = 0.3 M_\odot \ll M$$

   (b) (5 points) How long will it take the core to collapse to stellar dimensions? Give your answer in years. Again, state any assumptions that you make.

   Solution: The unstable core will soon be in free-fall collapse. Stellar size is so much smaller than 0.1 pc that we can take the time to be approximately the same as the free-fall time.

$$t_{ff} = \sqrt{\frac{3\pi}{32 G\rho}} = 5.2 \times 10^{12} \text{ s} = 1.7 \times 10^5 \text{ yr}$$

(c) (10 points) Suppose that we have found a clump with very little rotational motion so that the material all collapses into a spherical protostar which eventually becomes a $10M_\odot$ star. Estimate the radius of this star, and from this estimate and the previous results, estimate the average luminosity of the object during the collapse that forms the star.

Solution: In our experience with main-sequence stars, we learned that the radius scales roughly linearly with mass, so

$$\begin{aligned} R &\approx M = 10M_\odot \\ R &\approx 10R_\odot \end{aligned}$$

The luminosity of the star is a result of the gravitational potential energy of the star's particles converted into radiation:

$$\begin{aligned} \Delta E &= \frac{3GM^2}{5R} - \frac{3GM^2}{5r} = 2.3 \times 10^{49} \text{ erg} \\ L &= \frac{\Delta E}{\Delta t} = \frac{\Delta E}{t_{ff}} = 4.4 \times 10^{36} \text{ erg/s} = 1100 L_\odot \end{aligned}$$

6. A star with mass M and outer radius R has a density given as a function of radius r given by

$$\rho(r) = \rho_0 \left(1 - \frac{r^2}{R^2}\right)$$

(a) (10 points) Find an expression for the central density ρ_0 in terms of M and R.

Solution:

$$\begin{aligned}
M &= \int_0^R \rho(r) 4\pi r^2 \, dr \\
&= 4\pi\rho_0 \int_0^R r^2 \, dr - \frac{4\pi\rho_0}{R^2} \int_0^R r^4 \, dr \\
&= \frac{4\pi\rho_0}{3} R^3 - \frac{4\pi\rho_0}{5R^2} R^5 \\
&= \frac{8\pi\rho_0 R^3}{15}
\end{aligned}$$

$$\boxed{\rho_0 = \frac{15M}{8\pi R^3}}$$

(b) (20 points) Find an expression for the central pressure P_0 in terms of M and R.

Solution: Start with hydrostatic equilibrium:

$$P(R) - P_0 = \int_0^R \frac{dP}{dr} dr = -\int_0^R \frac{GM(r)\rho(r)}{r^2} dr$$
$$P_0 = \int_0^R \frac{GM(r)\rho(r)}{r^2} dr$$

Now we need an expression for the mass $M(r)$ contained within the radius r, which we get by integrating the density between 0 and r:

$$\begin{aligned} M(r) &= \int_0^r \rho(r')4\pi r'^2\, dr' = \frac{15M}{2R^3} \int_0^r \left(1 - \frac{r'^2}{R^2}\right) r'^2\, dr' \\ &= \frac{15M}{2} \int_0^{r/R} (1-u^2)u^2\, du = \frac{15M}{2} \left[\frac{u^3}{3} - \frac{u^5}{5}\right]_0^{r/R} \\ M(r) &= \frac{5M}{2} \left(\frac{r}{R}\right)^3 \left[1 - \frac{3}{5}\left(\frac{r}{R}\right)^2\right] \end{aligned}$$

Now plug this into the hydrostatic equilibrium integral above:

$$\begin{aligned} P_0 &= \int_0^R \frac{GM(r)\rho(r)}{r^2}\, dr = G\frac{15M}{8\pi R^3}\frac{5M}{2} \int_0^R \frac{1}{r^2}\left(\frac{r}{R}\right)^3 \left[1 - \frac{3}{5}\left(\frac{r}{R}\right)^5\right] \left[1 - \left(\frac{r}{R}\right)^2\right] dr \\ &= \frac{75GM^2}{16\pi R^4} \int_0^1 u \left(1 - \frac{3}{5}u^2\right)(1-u^2)\, du \end{aligned}$$

$$\boxed{P_0 = \frac{15GM^2}{16\pi R^4}}$$

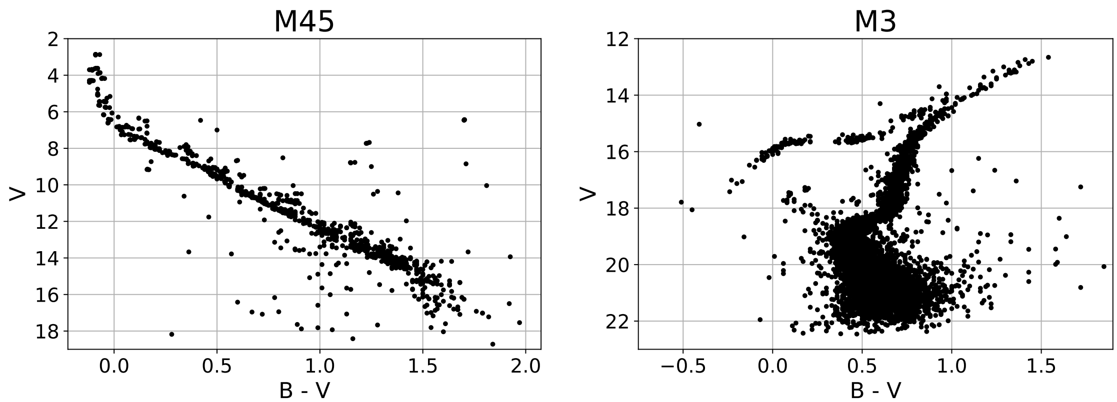
(c) (5 points) Find an expression for the temperature at the center of the star.

Solution: Rearrange the ideal gas law:

$$T(0) = \frac{\mu P_0}{\rho_0 k} = \frac{\mu}{k}\frac{8\pi R^3}{15M}\frac{15GM^2}{16\pi R^4}$$

$$\boxed{T(0) = \frac{GM\mu}{2kR}}$$

7. Below are H-R diagrams fro two star clusters, M45 (left) and M3 (right); the apparent V magnitude is plotted against $B - V = m_B - m_V$.



(a) (5 points) Rank the clusters in order of age. Roughly how old is the older one? (Use a sentence or two to explain each answer.)

> **Solution:** None of the members of M45 are very far from the main sequence, but there are many subgiants, red giants, horizontal-branch, and AGB stars in M3, and hardly any stars on the main sequence above the subgiants. Thus, M3 must be (much) older.
>
> The main-sequence turnoff in M3 takes place at $B - V \approx 0.4$. This is a bit warmer than the Sun; the corresponding stellar mass would be slightly larger than a solar mass as a result, and the total lifetime of such stars would be several billion years. Therefore, the cluster must be at least that old.

(b) (5 points) The Sun has an absolute V magnitude of $+4.83$ and a $B-V$ color index of 0.65. Use this information along with the H-R diagrams to estimate the distance to M45. (Neglect extinction, and express your answer in parsecs.)

Solution: The apparent magnitude of members of M45 that have $B-V=0.65$ is about 10.3. Such stars can be presumed to be just like the Sun, so the distance is

$$m = M + 5\log\left(\frac{r}{10\ \text{pc}}\right)$$
$$r = (10\ \text{pc})10^{(m-M)/5} = (10\ \text{pc})10^{(10.3-4.83)/5}$$
$$r = 130\ \text{pc}$$

(c) (5 points) The nearby star RR Lyrae has an average, absolute V magnitude of $+0.75$ and a distance of 250 pc. Estimate the distance to M3.

Solution: If RR Lyr is a standard candle, then RR Lyr stars in M3 will have the same absolute magnitude as RR Lyr itself:

$$M_{RRL} = M_{M3} = m_{M3} - 5\log\left(\frac{r_{M3}}{10\ \text{pc}}\right)$$
$$r_{M3} = (10\ \text{pc})10^{(m_{M3}-M_{RRL})/5}$$

The RR Lyr stars in M3 would likely be in the gap in the horizontal branch, which would give them an average apparent magnitude of about 15.5. Therefore,

$$r_{M3} = (10\ \text{pc})10^{(15.5-0.75)/5}$$
$$\boxed{r_{M3} = 8.9\ \text{kpc}}$$