Astronomy 465 — Problem Set 4

Prof. Kelly Douglass

Due Tuesday, November ~~1~~ 3 at 2PM EDT

1. Show that the total mass of an infinitesimally-thin, exponential disk with a surface density distribution of
$$\Sigma(R) = \Sigma_0 e^{-R/R_d}$$
is
$$M_d = 2\pi\Sigma_0 R_d^2$$

2. Show that the gravitational potential of an infinitesimally-thin exponential disk with a surface mass density of
$$\Sigma(R) = \Sigma_0 e^{-R/R_d}$$
is
$$\Phi(R, z) = -2\pi G\Sigma_0 R_d^2 \int_0^\infty \frac{J_0(kR)e^{-k|z|}}{[1 + (kR_d)^2]^{3/2}} dk$$

3. *Orbits in disk galaxies*

   (a) Show that the equations of motion for orbits in an axisymmetric potential $\Phi(r, z)$ that is symmetric about the plane $z = 0$ are
$$\ddot{r} - r\dot{\phi}^2 = -\frac{\partial\Phi}{\partial r}$$
$$\frac{d}{dt}(r^2\dot{\phi}) = 0$$
$$\ddot{z} = -\frac{\partial\Phi}{\partial z}$$
   (Hint: What is the Lagrangian of the orbit?)

   (b) The second of the above equations represents the conservation of angular momentum about the z-axis, $L_z = r^2\dot{\phi}$, while the other two describe coupled oscillations in the radial and z directions. Show that these equations can be simplified to
$$\ddot{r} = -\frac{\partial\Phi_{\text{eff}}}{\partial r} \qquad \ddot{z} = -\frac{\partial\Phi_{\text{eff}}}{\partial z}$$
   where
$$\Phi_{\text{eff}}(r, z) = \Phi(r, z) + \frac{L_z^2}{2r^2}$$
   The orbits are therefore in the **meridional plane** (the $(r,\ z)$ plane), which rotates about the symmetry axis with $\dot{\phi} = \frac{L_z}{r^2}$.

   (c) Plot the level contours of the effective potential, Φ_{eff}, in the (r, z) plane when
$$\Phi(r, z) = \frac{1}{2}v_0^2 \ln\left(r^2 + \frac{z^2}{q^2}\right)$$
   where $v_0 = 1$, $L_z = 0.1, 0.5$, and $q = 0.9$. For what value of L_z will orbits actually exist on the symmetry axis?

(d) Where does the minimum of the effective potential, Φ_{eff}, occur? This is known as the **guiding center**.

(e) Expand Φ_{eff} in a Taylor series around $(x = r - R_g, z) = (0, 0)$ to show that the equations of motion in the meridional plane become

$$\ddot{x} = -\kappa^2 x \qquad \ddot{z} = -\nu^2 z$$

where $\kappa^2 \equiv \left(\frac{\partial^2 \Phi_{\text{eff}}}{\partial r^2}\right)_{(R_g,0)}$ is the epicyclic frequency, and $\nu^2 \equiv \left(\frac{\partial^2 \Phi_{\text{eff}}}{\partial z^2}\right)_{(R_g,0)}$ is the vertical frequency.

4. *Spiral structure in the Milky Way.* Figure 1 contains reproduced HI 21 cm line observations obtained with the Dwingeloo radio telescope when successively directed to points along the galactic equator at intervals of 5° in galactic longitude (van de Hulst et al. 1954). The value of Galactic longitude (in degrees) corresponding to each spectrum is obtained by adding 33 to the number with which it is labeled in Figure 1, subtracting 360 if the result exceeds 360. From the horizontal scale in Figure 1, the frequency shift and the radial velocity with respect to the local standard of rest is directly obtained.

   Determine the Galactocentric distances of the diffuse clouds represented by peaks in the 21 cm line spectra (Figure 1) and plot the distances as a function of angular position to reveal the spiral structure of atomic gas in the Milky Way. Assume that $r_\odot = 8.4$ kpc and $\Omega(r_\odot) = 9.8 \times 10^{-16}$ rad/s = 30 km/s/kpc.

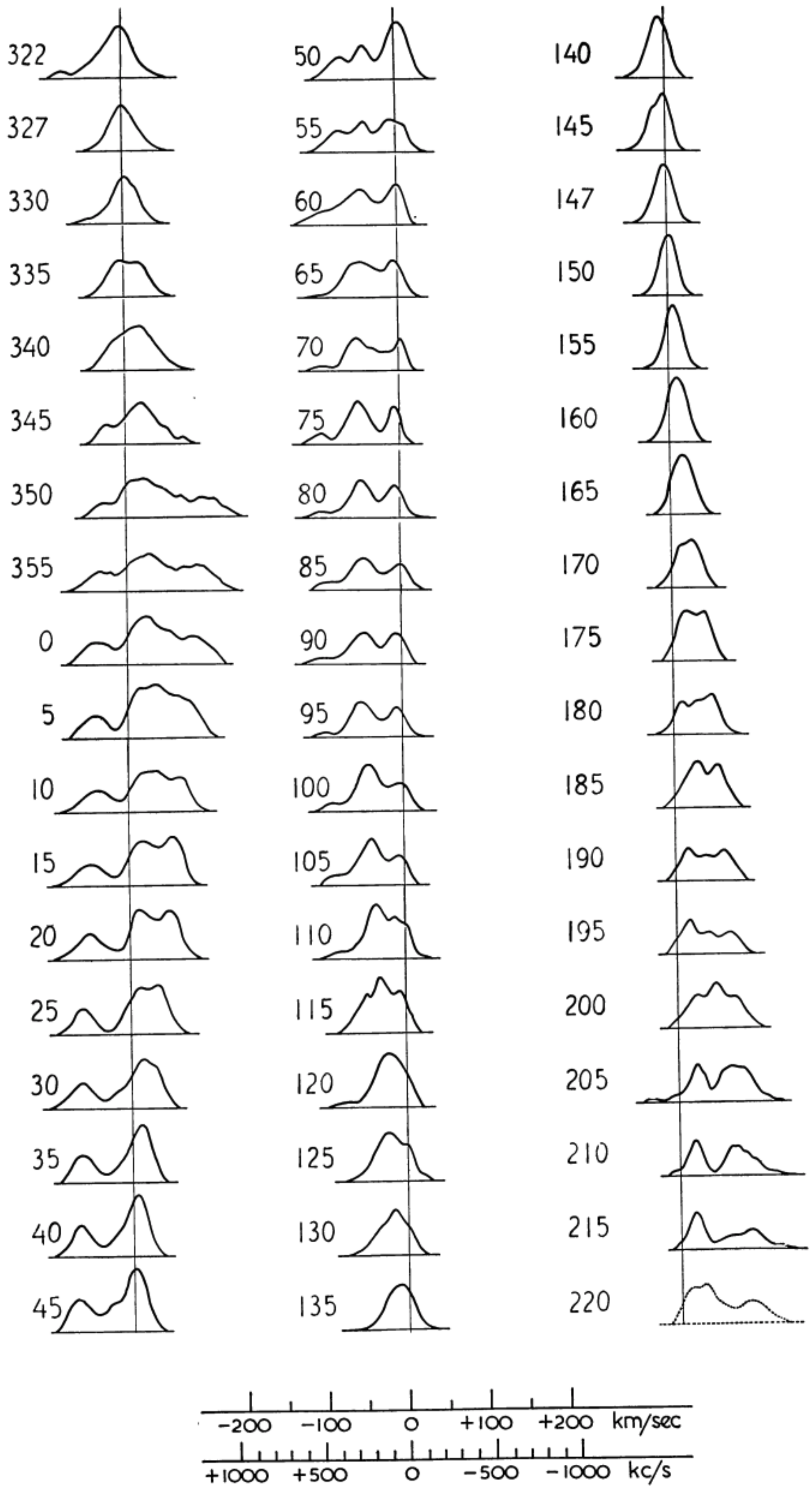
   a. For $\ell = 8° - 78°$, find the HI peak with the largest redshift; assume that this peak represents hydrogen at the tangent point, deduce v_r, r, $v_r/r = \Omega(r) - \Omega(r_\odot)$, and then $\Omega(r)$. The radial velocities can be measured on the figure with a ruler.

   b. For the other peaks in each (or every other) longitude interval, you are now in a position to consecutively calculate v_r, $\sin \ell$, $\Omega(r) - \Omega(r_\odot)$, r, and then, using the law of sines, θ (see Figure 2). Begin with the first (left) maximum of the profile at $\ell = 88°$. This can done rather conveniently in Excel; remember that the Excel trig functions expect the arguments to be in radians.

   c. Plot each cloud in a polar diagram representing the galactic plane. (In such plots, the Sun is usually drawn above the Galactic center so that the $\ell = 0$ axis points down.) Remember that the distance r is measured *from the Galactic center*. You may find it helpful to do this plot in Cartesian coordinates, which in the layout of Figure 2 would be $x = r \sin \theta$ and $y = r \cos \theta$.

   d. When r is found to be smaller than $r_\odot$, the distance ambiguity applies. This ambiguity cannot be solved without additional data and consideration. Therefore, limit your investigation to the region outside the solar orbit, with the exception of the tangent points. (You have already completed one side.) This means that for $0 < \ell < 180°$, use only the peaks for which $v_r < 0$; and for $180° < \ell < 360°$, use only the peaks for which $v_r > 0$.

   e. Connect the plotted positions of the hydrogen clouds by smooth lines, in so far as continuity is suggested by the successive profiles. Compare your results to the cartoon of the Galaxy's structure as is currently known (from the lecture notes). Which spiral features do you detect?



Survey of line profiles at various longitudes.

Figure 1: Profiles of the hydrogen emission line at 21 cm, from van de Hulst et al. (1954).

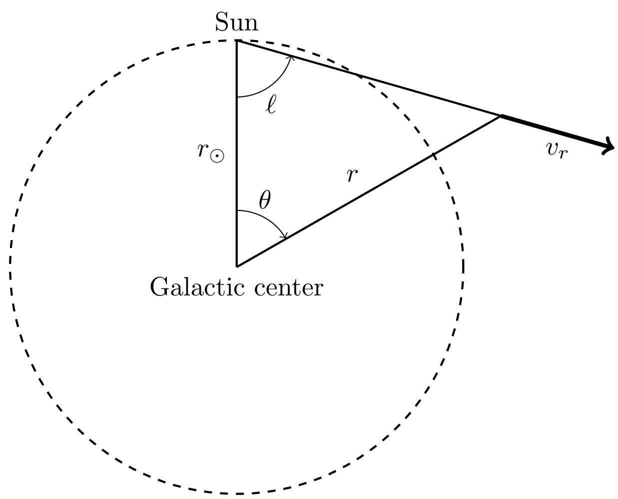


Figure 2: Geometry of longitude and LSR radial velocity measurements. The dotted circle is the orbit of the LSR about the Galactic center.