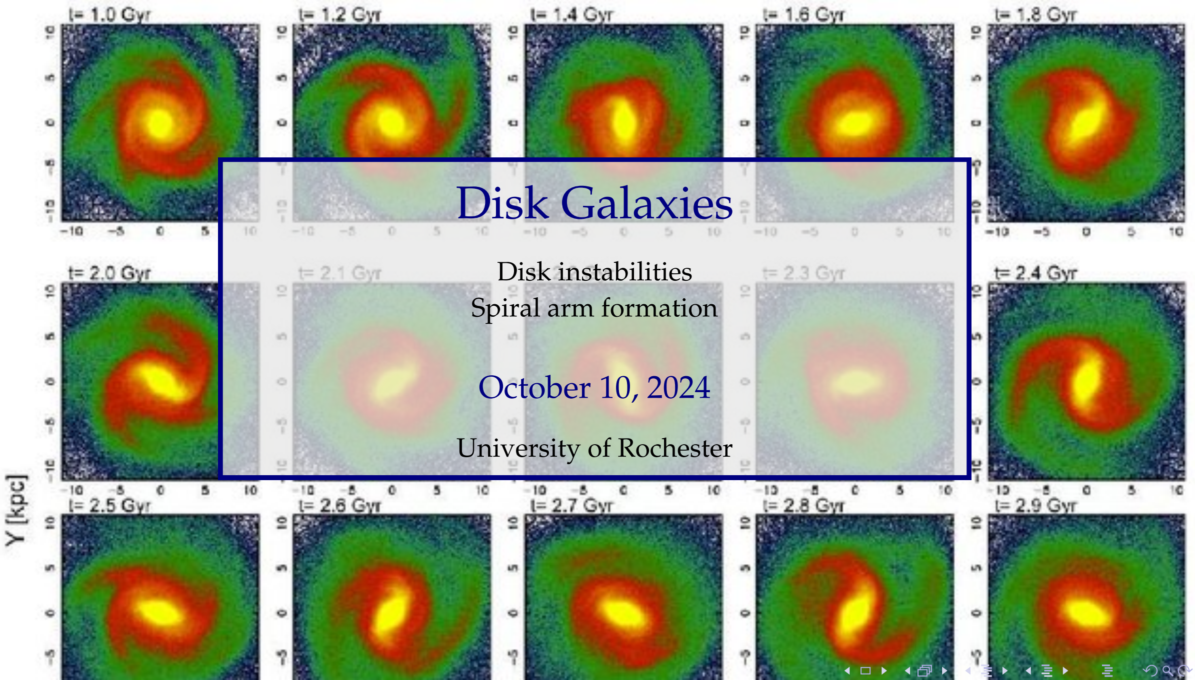

Disk Galaxies

Disk instabilities
Spiral arm formation

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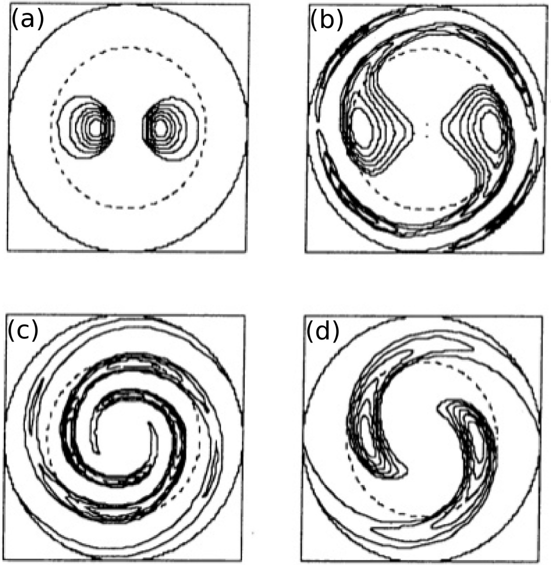
Disk instabilities

Global instabilities can cause a significant transformation of the overall disk structure.

Local instabilities determine if perturbations smaller than the disk size can grow.



Density contours of global instability nodes for a rotating disk (Dobbs & Baba, 2014)

Local instabilities

Local instabilities occur when the characteristic size of the perturbation is much smaller than the disk.

If we write the perturbation mode as

$$\Sigma_1(r,\phi,t) = A(r,t)e^{i[m\phi+f(r,t)]}$$

where

$f(r,t)$ is the shape function

$A(r,t)$ is the amplitude of the density wave that varies slowly in r

then, for fixed t, $m\phi + f(r,t) =$ constant, and peaks in the density waves occur when $m\phi + f(r,t) = 2\pi n$.

Local instabilities: Tightly-wound waves

A perturbation in the neighborhood of (R_0, ϕ_0) is then

$$\Sigma_1(r, \phi, t) \approx \Sigma_a e^{ik(R_0,t)(r-R_0)}$$

where

$$\begin{aligned} \Sigma_a &= A(R_0, t) e^{i(m\phi_0 + f(R_0,t))} \\ k(R_0, t) &\equiv \left[\frac{\partial f(r,t)}{\partial r}\right]_{R_0} = \frac{2\pi}{\Delta r} \end{aligned}$$

Tightly-wound waves are essentially plane waves in the radial direction, with a wave-vector $k\hat{r}$ and wavelength Δr.

Local instabilities in a gaseous disk

If we define the dimensionless parameters

$$Q \equiv \frac{c_s \kappa}{\pi G \Sigma_0} \qquad \lambda_{\rm crit} \equiv \frac{4\pi^2 G \Sigma_0}{\kappa^2}$$

then the dispersion relation becomes

$$\omega^2 = \frac{4\pi^2 G \Sigma_0}{\lambda_{\rm crit}} \left[1 - \frac{\lambda_{\rm crit}}{\lambda} + \frac{Q^2}{4} \left(\frac{\lambda_{\rm crit}}{\lambda} \right)^2 \right]$$

where $\lambda \equiv \frac{2\pi}{|k|}$.

The limiting value of Q is

$$Q(\lambda) = 2\sqrt{\frac{\lambda}{\lambda_{\rm crit}} \left(1 - \frac{\lambda}{\lambda_{\rm crit}} \right)}$$

Local instabilities in a stellar disk

With the pressure a result of random stellar motion, the dispersion relation is

$$(m\Omega - \omega)^2 = \kappa^2 - 2\pi G\Sigma_0 |k| F\left(\frac{\omega - m\Omega}{\kappa}, \frac{k^2\sigma_r^2}{\kappa^2}\right)$$

where

$$\begin{aligned} \sigma_r &= \text{radial velocity dispersion} \\ F(s,x) &= \tfrac{1-s^2}{\sin(s\pi)} \int_0^\pi e^{-(1+\cos\tau)x} \sin(s\tau)\sin\tau\, d\tau \end{aligned}$$

While λ's definition does not change, $Q \equiv \frac{\sigma_r\kappa}{\pi G\Sigma_0}$.

In the limiting case ($\omega = 0$),

$$\frac{\lambda_{\text{crit}}}{\lambda} F\left(0, Q^2\left(\frac{\lambda_{\text{crit}}}{\lambda}\right)^2\right) = 1$$

Toomre stability criterion

The limiting inequalities

$$Q \equiv \frac{c_s \kappa}{\pi G \Sigma_0} > 1 \qquad \text{(stable gaseous disk)}$$
$$Q \equiv \frac{\sigma_r \kappa}{3.36 G \Sigma_0} > 1 \qquad \text{(stable stellar disk)}$$

are known as Toomre's instability criterion, with Q sometimes called the Toomre parameter.

Note: The Toomre instability criterion is derived for axisymmetric perturbations with a wavelength much smaller than the disk size.

Global instability in the Maclaurin disk

The Maclauren disk is a fluid disk with an unperturbed surface density equal to

$$\Sigma_0(r) = \begin{cases} \Sigma_c \sqrt{1 - \left(\frac{r}{a}\right)^2} & r \leq a \\ 0 & r > a \end{cases}$$

where a is the radius of the disk. The corresponding potential in the disk is

$$\Phi_0(r) = \frac{1}{2}\Omega_0^2 r^2 + \text{constant} \qquad \Omega_0^2 = \frac{\pi^2 G \Sigma_c}{2a}$$

The dispersion relation is then

$$(\omega - m\Omega)^3 - (\omega - m\Omega)\left\{4\Omega^2 + (\ell^2 + \ell - m)\left[\Omega_0^2(1 - g_{\ell m}) - \Omega^2\right]\right\}$$
$$+ 2m\Omega\left[\Omega_0^2(1 - g_{\ell m}) - \Omega^2\right] = 0$$

where $g_{\ell m} = \frac{(\ell+m)!(\ell-m)!}{2^{2\ell-1}\left[\left(\frac{\ell+m}{2}\right)!\left(\frac{\ell-m}{2}\right)!\right]^2}$ and $0 \leq m \leq \ell$ are integers.

Global instability in the Kalnajs disk

The Kalnajs disk is a stellar disk with an unperturbed surface density and potential equal to that of the Maclauren disk.

The pressure is provided by the random motion of the stars, where

$$\langle v_x^2 \rangle = \langle v_y^2 \rangle = \frac{a^2}{3}(\Omega_0^2 - \Omega^2)\left(1 - \frac{r^2}{a^2}\right)$$

The disk becomes unstable when

$$\left(\frac{\Omega}{\Omega_0}\right)^2 < \frac{125}{486}$$

Do galaxies exhibit bar instabilities?

Kinematic observations of the Milky Way indicate that the disk should be unstable to the bar mode, but the Milky Way disk is stable.

Alternate expression for the bar instability (based on observables):

$$\begin{aligned} \varepsilon_m \equiv V_{\max}\sqrt{\frac{R_d}{GM_d}} &\gtrsim 1.1 \qquad \text{(stable stellar disks)} \\ &\gtrsim 0.9 \qquad \text{(stable gaseous disks)} \end{aligned}$$

An isolated exponential disk has $\varepsilon_m \approx 0.63$.