Kevin McFarland
Physics 254, Fall '00

Relativistic Kinematics

Notation and Four Vectors

For this class we will always use 4-vector notation when discussing relativistic kinematics, e.g.

$$\boldsymbol{p} = (p_0, p_1, p_2, p_3) = (p_0, \vec{p}),$$

where $\vec{p}$ is a three-space vector. For two 4-vectors, $\boldsymbol{p}$ and $\boldsymbol{q}$, the inner ("dot") product is defined as

$$\boldsymbol{p} \cdot \boldsymbol{q} \equiv g_{\mu\nu} p_\mu q_\nu,$$

where the non-zero components of the metric tensor $g_{\mu\nu}$ are $g_{00} = 1$, $g_{11} = g_{22} = g_{33} = -1$. In other words,

$$\boldsymbol{p} \cdot \boldsymbol{q} = p_0 q_0 - \vec{p} \cdot \vec{q}.$$

This notation is what you are likely to encounter most often in standard introductory field theory texts, e.g., Bjorken and Drell, Mandl and Shaw, Peskin and Schroeder. Note that the sum over the repeated indices above (μ and ν) is implied, and that you will often see the notation $p^\nu q_\nu$ as a shorthand for $g_{\mu\nu} p_\mu q_\nu$, in other words $p^\nu = g^{\mu\nu} p_\mu$, where $g^{\mu\nu} g_{\nu\lambda} = \delta^\mu_\lambda$, the usual Kronecker delta.

Welcome to the big leagues.

Lorentz Transformations

Let the energy and three-momentum in a particular reference frame, A, particle be represented as the 4-vector, $\boldsymbol{p_A}$,

$$\boldsymbol{p_A} = (E_A, \vec{p_A}).$$

Viewed from a frame, B, moving at velocity β with respect to frame A, the Lorentz transform

$$\begin{pmatrix} E_B \\ \vec{p_B} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\vec{\beta} \\ -\gamma\vec{\beta} & \gamma \end{pmatrix} \begin{pmatrix} E_A \\ \vec{p_A} \end{pmatrix},$$

where $\gamma = \frac{1}{\sqrt{1-\vec{\beta}^2}}$, gives the correct expression for the energy and momentum 4-vector as viewed from frame B, $\boldsymbol{p_B}$.

Lorentz Scalars

The inner product of two 4-momenta, $\boldsymbol{p}\cdot\boldsymbol{q}$ is invariant under a Lorentz transformation. A particular case is the 4-momentum squared of a free particle,

$$\boldsymbol{p}^2 = E^0 - \vec{p}^2 = m^2,$$

where m is the rest mass. If a 4-momentum has $\boldsymbol{p}^2 > 0$, e.g., a free particle, it is referred to as "timelike", and if a 4-momentum has $\boldsymbol{p}^2 < 0$, it is "spacelike". If $\boldsymbol{p}^2 = 0$, the 4-momentum is sometimes referred to as "lightlike" or "on the light cone".

If the 4-momentum sqaured of a particle involved in an interaction is equal to its mass squared, the particle is often referred to as "real" or "on mass shell" or just "on-shell". If this is not true, the particle is "virtual" or "off-shell". Virtual particles can only exist for short times *via* the grace of the uncertainty principle. In other words, all particles in the final or initial state of a reaction will be "on-shell".

Lorentz-Invariant "Phase Space"

When calculating rates for, e.g., particle production, the density of available plane wave states ("phase-space"), $d^3\vec{p}$ often enters. Unfortunately, this is not a Lorentz invariant quantity, as can be easily seen from the transformation above.

Consider a Lorentz boost along the $\hat{z}$ axis, so that $\vec{\beta} = \beta\hat{z}$. Then

$$p'_z = \gamma(p_z - \beta E),$$

and

$$E' = \gamma(E - \beta p_z).$$

Then,

$$dp'_z = \frac{\partial p'_z}{\partial p_z} dp_z + \frac{\partial p'_z}{\partial E} E$$

$$
\begin{aligned}
&= \gamma(1-\beta\frac{p_z}{E})dp_z \\
&= \frac{E'}{E}dp_z.
\end{aligned}
$$

So,

$$\frac{dp'_z}{E'} = \frac{dp_z}{E},$$

and therefore,

$$\frac{d^3\vec{p}'}{E'} = \frac{d^3\vec{p}}{E}.$$

However, the choice of the $\hat{z}$ axis was arbitrary, so this result must hold for any boost.

Consider as an application the calculation of the cross-section for single particle production, which can be written

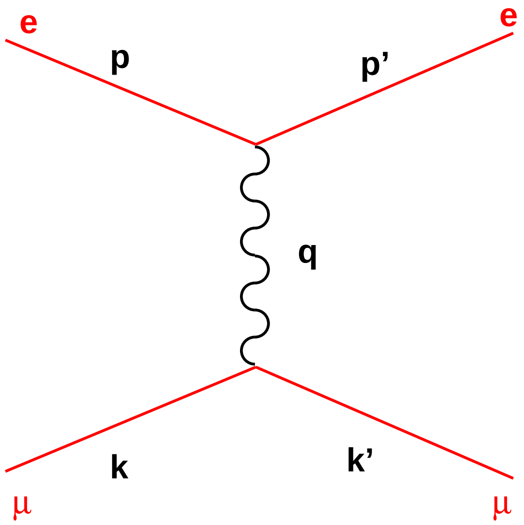
$$\frac{d^3\sigma}{d^3\vec{p}}.$$

This quantity is not Lorentz invariant, but a related quantity,

$$\frac{Ed^3\sigma}{d^3\vec{p}}.$$

is, by the above argument. This quantity is referred to as a "Lorentz invariant cross-section".

Mandelstam Invariants



Consider the scattering process $e^-\mu^- \to e^-\mu^-$ shown in the Feynman diagram above. Three Lorentz invariants that can be constructed to describe the scattering process are

$$\begin{aligned} s &= (\boldsymbol{k}+\boldsymbol{p})^2 = (\boldsymbol{k'}+\boldsymbol{p'})^2, \\ t &= (\boldsymbol{k}-\boldsymbol{k'})^2 = (\boldsymbol{p}-\boldsymbol{p'})^2, \\ u &= (\boldsymbol{p}-\boldsymbol{k'})^2 = (\boldsymbol{k}-\boldsymbol{p'})^2. \end{aligned}$$

These are the so-called Mandelstam variables. Each of them is a Lorentz invariant, and they are related by the identity

$$s+t+u = \sum m^2,$$

where the sum is over the initial and final state particles in this $2 \to 2$ scattering process.

Two of the Mandelstam variables have special meaning in this scattering process. s is the 4-momentum squared of the initial state. In the center-of-mass frame, where the total momentum of the initial state is zero,

$$s = (p_0 + k_0)^2,$$

so that $\sqrt{s}$ is the total energy in the center-of-mass frame or the "center-of-mass energy". (Often you will hear physicists working at an accelerator speak of the "root s" as a shorthand for center-of-mass energy.) Conservation of energy tells us that this is the total amount of energy which can be observed in the final state of a reaction as well.

t is the square of the 4-momentum carried by the boson mediating the scattering process above, and represents the square of the 4-momentum transferred from the muon to the electron to the other. t is given by

$$\begin{aligned} t = (\boldsymbol{k} - \boldsymbol{k'}) &= \boldsymbol{k}^2 + (\boldsymbol{k'})^2 - 2\boldsymbol{k} \cdot \boldsymbol{k'} \\ &= 2m_\mu^2 - 2k_0 k_0' + 2\vec{k} \cdot \vec{k'}. \end{aligned}$$

In the ultra-relativistic limit where masses are negligible, this becomes

$$t \approx -2k_0 k_0'(1 - cos\theta),$$

where θ is the scattering angle in the reaction. Note that t is always negative ("spacelike") and that $t \to 0$ when $\theta \to 0$, in other words when the scattering is negligible. Thus a large, negative t indicates a significant scatter. Note also that this implies that the intermediate particle in the reaction (the carrier of the scattering force) is virtual since its 4-momentum is spacelike and becomes more virtual as the scattering angle increases.