E&M Cheat Book

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1 Introduction

These notes are meant for the introductory honors electricity and magnetism course offered at the University of Rochester (PHY142). It offers the same (or similar) summary at the end of every chapter as the on in O&M. It might be useful to the student to have those summaries condensed into one document. I have included some tips and tricks in order to highlight what is important to take away from the class, and E&M at the introductory level in general. This document assumes the reader is fluent in vector calculus and other necessary maths.

Other textbooks used are D.J. Griffith's Introduction to Electrodynamics. This book is typically used by the higher level E&M courses, covering E&M I & II (P217/218). Another source that might be useful is Purcell's Electricity and Magnetism, which is at a similar level. The book typically used in a graduate course is Classical Electrodynamics by Jackson. Lots of Green's functions.

It is obvious based on the title of textbooks on the subject, that there is an order to learning E&M. First one starts off with electrostatics and magnetostatics. Perhaps covering some of its applications, especially when it comes to circuits. Then one moves on from "statics," where there are no moving charges, to those that are moving, or "dynamics." The student learns Maxwell's equations and learns that there are four of them and that they can be written in differential and integral form. This typically marks the end of the undergraduate "freshman" level of the subject - or this course.

- M.B. Adams (09/04/2013)

2 Electrostatics

Electrostatics first starts off by discussing Coulomb's law, the fundamental equation of electrostatics, which describes the force between two charges. It is important to note the similarity in mathematical expression between the Coulomb force and gravitational force. The electric field is generated by these charges (as well as a magnetic field). There are a variety of ways of determining the electric field due to a charge, or charge distribution. We can also talk about the energy of these charges playing in the field.

From this a few important equations come about:

$$\begin{cases} \mathbf{F_{12}} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{21}^2}\hat{\mathbf{r}} & \text{Coulomb's Law} \\ \mathbf{F} = q\mathbf{E} & \text{General Equation for Electric Force} \\ \nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}, \oint E_{\perp}dA = \frac{Q_{encl}}{\epsilon_0} & \text{Gauss' Law (Maxwell Eq. 1)} \\ \nabla\times\mathbf{E} = \mathbf{0} & \text{Proof we're dealing with statics! (Maxwell's Eq. 2)} \\ \mathbf{E} = -\nabla V & \text{Getting E using V} \\ V = -\int \mathbf{E}\cdot d\mathbf{l} & \text{Getting V using E} \end{cases}$$

2.1 Electric Force and Electric Charge

- **Electric charges** may be positive, negative, or zero; like charges repel, unlike charges attract. The **SI unit of charge** is 1 coulomb = $1C$

- **"Fundamental Charge," or charge of a proton** is $e = 1.6 \times 10^{-19} C$, the **charge of an electron** is the same, but negative, $-e = 1.6 \times 10^{-19} C$.

- **Coulomb's Law** states that the direction of the Coulomb force is along the line joining the particles. It is stated as $\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{Q_1 Q_2}{r_{12}} \hat{\mathbf{r}}$. The **Permittivity constant** (electric constant) is known to be $\epsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{Nm^2}$. Sometimes you'll see Coulomb's law written with the **Coulomb constant**, $k = \frac{1}{4\pi\epsilon_0} = 8.99 \times 10^9 \frac{Nm^2}{C^2}$.

- **Superposition Principle** is another name for the net force, or vector sum of individual forces.
  i.e. $\mathbf{F}_{net} = \sum_{i=1}^{N} \mathbf{F}_i = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + ... + \mathbf{F}_N$.

- **Charge Conservation**, or that in any reaction or process, the net electric charge remains constant.

- **Charge Quantization**, or any charge should be thought of as an integer multiple of the fundamental charge.

- An **ion** is an atom with net charge - missing or extra atoms.

- **Electrolyte** is a liquid with many dissolved ions.

- **Plasma** is a gas with many ionized atoms and free electrons.

- A **conductor** permits the motion of charge, whereas an **insulator** does not.

2.2 The Electric Field

- Let's revisit the **superposition principle**. Electric forces and electric fields produced by different charges or by different charge distributions combine over vector addition.

- We can define the **electric field** to be, $\mathbf{E} = \frac{\mathbf{F}}{q}$. We can figure out the **SI unit of an electric field based on this equation**, $\frac{1N}{1C}$.

- The **electric field of a point charge** q is $|\mathbf{E}| = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}$. The electric field is a vector quantity, so the direction in this case would be radially outward for a positive charge, and radially inward for a negative charge.

In either case if we were to write the electric field using the unit vector notation, we'd say $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$.

- **Electric field of a continuous charge distribution:** Given we can describe the total field $\mathbf{E} = E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}}$. This is obtained by summing contributions to each compotent, i.e. $E_x = \int dE_x$ where $dE_x = \cos\theta dE = \frac{1}{4\pi\epsilon_0}\cos\theta\frac{dq}{r^2}$. Where θ is the angle between the electric field contribution and the x-axis.

$$dE = \frac{1}{4\pi\epsilon_0}\frac{dq}{r^2} \Longrightarrow E = \int dE = \frac{1}{4\pi\epsilon_0}\int\frac{dq}{r^2}$$

- **Charge distributions** are things we're going to have to become more use to. They can come in a **linear** or **surface** form.

  1. Linear: $dq = \lambda dL$, where λ is in coulomb's per meter.
  2. Surface: $dq = \sigma dA$, where σ is in coulomb's per square meter.

- The **electric field of an infinite, uniformly charged thin rod** is $E = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r}$, where the direction is perpendicular to the road (outward for $+\lambda$, inward for $-\lambda$.

- The **electric field of an infinite, uniformly charged flat sheet** is $E = \frac{\sigma}{2\epsilon_0}$, where the direction is perpendicular to the sheet (outward for $+\sigma$, inward for $-\sigma$.

- What if we want to know the **electric field of a pair of oppositely charged, parallel flat sheets**?

$$E = \begin{cases} \frac{\sigma}{\epsilon_0} & \text{(between the sheets)} \\ 0 & \text{(outside of the sheets)} \end{cases}$$

- **Properties of electric field lines:**

  1. Lines are tangent to the electric field vector at any point.
  2. Density of lines is proportional to the magnitude of the field.
  3. Field lines do not cross.

4. Field lines start on positive charges and end on negative charges.
5. The number of field lines emerging from (terminating on) a positive (negative) charge is proportional to the charge.

- **Motion in uniform E**:

$$\mathbf{a} = \frac{\mathbf{F}}{m} = \frac{q\mathbf{E}}{m}$$

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{a}t$$

$$\mathbf{r} - \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2$$

- The **electric dipole moment** is defined as $p = \ell Q$, where the direction of the dipole moment vector $\mathbf{p}$ is from $-$ to $+$.
- The **torque felt by a dipole** is defined to be $\tau = \mathbf{p} \times \mathbf{E} = -pE\sin\theta$.
- A dipole also feels a **potential energy**, $U = -pE\cos\theta = -\mathbf{p} \cdot \mathbf{E}$.

2.3 Gauss' Law

- The **electric flux through an open surface** is defined by $\Phi_E = \mathbf{E} \cdot A = E_\perp A = EA\cos\theta$ for a flat surface, with uniform $\mathbf{E}$. Is we're working with an arbitrary surface and varying $\mathbf{E}$, we integrate,

$$\Phi_E = \int \mathbf{E} \cdot d\mathbf{A} = \int E_\perp dA.$$

- The **electric flux through a closed surface** for a positive–outward $\mathbf{E}$ is

$$\oint \mathbf{E} \cdot d\mathbf{A} = \oint E_\perp dA.$$

- **Gauss' Law** states that the electric flux through a closed surface area is equal to the charge enclosed by the volume of the object over the permittivity constant.

$$\oint E_{\perp} dA = \frac{Q_{enclosed}}{\epsilon_0}$$

where

$$\oint E_{\perp} dA \;=\; \begin{cases} E(4\pi r^2) & \text{(spherical)} \\ E(2\pi r L) & \text{(cylindrical)} \\ 2EA & \text{(planar, both ends in the field)} \\ EA & \text{(planar, one end in the field)} \end{cases} .$$

Make sure to consider the symmetry of the object you're trying to determine the electric field for. There are only a few coordinate systems we know of, so it will probably involve one of those.

- For line, surface or volume charge, one must know of their **uniform charge distribution**. They are $q = \lambda L$, $q = \sigma A$, and $q = \rho V$ respectively.

- It is important to note that the electric field within a conducting material is zero. Any charge resides on the surface(s). The electric field at the surface is perpendicular and of magnitude $E = \frac{\sigma}{\epsilon_0}$.

2.4 Electrostatic Potential and Energy

- The **electrostatic potential, or potential energy per unit charge** is defined as, $V = \frac{U}{q}$. The **SI unit of potential** is 1 volt = 1 V = 1 J/C. An alternate unit of energy is the electron volt, 1 eV = 1.60×10^{-19} J.

  - When working with electron volts, remember that you're actually multiplying the charge of the electron, e, by the voltage. One can forget this sometimes while doing problems.

- If an electric field is uniform, it is clear that the **electrostatic potential in a uniform electric field** E_0 is linear, $V = -E_0 y$, where y is the distance along the field direction.

- Now that we're talking about energy - we need to talk about how it is conserved! **Conservation of energy** tells us that the total energy is defined as,

$$K + U = \frac{1}{2}mv^2 + qV = [constant].$$

  - As an aside, what you'll learn in P235 is that the above equation is called the Hamiltonian, H, of a charge, q, moving with a velocity, v, in a potential, V. The Hamiltonian is generally defined to be $H = T + U$. By the way, in P235 they denote kinetic energy by T.
  - p.s. The Lagrangian, L, which you might have been told about by some pretentious upperclassman (apparently that is me in this case) is generally defined to be $L = T - U$.

- The **potential energy of two point charges** is defined as

$$U = \frac{1}{4\pi\epsilon_0}\frac{qq'}{r}$$

- The **potential energy of a point charge** is defined as

$$U = \frac{1}{4\pi\epsilon_0}\frac{q'}{r}$$

- The above equations make sense because we can preform a calculation of the potential energy from the electric field! If V_0 is the potential at a point P_0 then at a point P,

$$V = -\int_{P_0}^{P} \mathbf{E} \cdot d\mathbf{s} + V_0$$

or more generally, for a continuous charge distribution,

$$V = \frac{1}{4\pi\epsilon_0} \int \frac{dQ}{r}.$$

- A note about **potential and conductors**: For static charge distributions, the potential throughout a conductor is constant. Within an empty cavity in a conductor, the electric field is exactly zero.

- Calculation of the **electric field from the potential** is simply $E = -\nabla V$, or the negative gradient of the potential.

- An **equipotential surface** is an imaginary surface on which the electrostatic potential is constant. The electric field is everywhere perpendicular to an equipotential surface.

- The **potential energy of a system of point charges is**

$$U = \frac{1}{4\pi\epsilon_0} \left[\frac{q_1 q_2}{r_{12}} + \frac{q_2 q_3}{r_{23}} + \frac{q_1 q_3}{r_{13}} + ... + [\text{all pairs}] \right].$$

- The **potential energy of a conductor** is defined to be $U = \frac{1}{2}QV$. If one is dealing with a system of conductors, simply sum over all $Q_i V_i$.

- The **energy density in an electric field** is defined as $u = \frac{1}{2}\epsilon_0 E^2$.

3 Applications of Electrostatics

3.1 Capacitors and Dielectrics

- The **SI unit of capacitance** is 1 F = 1 farad = 1 coulomb/volt. The **capacitance of a single conductor** is $C = \frac{Q}{V}$, for a **pair of conductors** it is $C = \frac{Q}{\delta V}$.

- The capacitance for specific geometries involve surface area.

  - For instance, the **capacitance of an isolated sphere** is $C = 4\pi\epsilon_0 R$.
  - The **capacitance of parallel plates** is $C = \frac{\epsilon_0 A}{d}$.

- Turns out parallel plate capacitors are things that get put on wires! Woo! We can talk about capacitors in circuit systems.
  - Capacitors in parallel have the same potential difference across each. So the **parallel combination of capacitors** is additive, or,
$$C = C_1 + C_2 + C_3 + ...$$
  - Capacitors in series have the same charge on each. The **series combination of capacitors** is defined as,
$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + ...$$
- The **capacitance per unit length of cylindrical capacitor** is
$$\frac{C}{\ell} = \frac{2\pi\epsilon_0}{\ln(b/a)}.$$
- The **electric field in a dielectric between parallel plates** is
$$E = \frac{1}{\kappa}E_{free}.$$
where κ is the dielectric constant. It is larger than 1. The relation $E = E_{free}/\kappa$ also applies in any dielectric with the same symmetry as the distribution of free charges.
- **Capacitance with dielectric** is simply $C = \kappa C_0$.
- The **energy stored in capacitor** can be written a variety of ways, as you can see:
$$U = \frac{1}{2}Q\delta V = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}C(\delta V)^2.$$
- The **energy density of a dielectric** is $u = \frac{1}{2}\kappa\epsilon_0 E^2$.

3.2 Direct Current (DC) Circuits

- The **electric field in uniform wire of length** ℓ is $E = \frac{\delta V}{\ell}$.
- The **electric current, or flow of charge per unit time** is defined as $I = \frac{dQ}{dt}$. The **SI unit of electric current** is 1 ampere = 1 A = 1 C/s.
- The **drift velocity (average velocity) in terms of the average collision time,** τ is defined as $v_d = \frac{-eE\tau}{m_e}$.
- The **current of free electrons in a conductor** is defined as $I = -nev_dA$, where n is the number density of free electrons, v_d is the drift velocity as stated above, and A is the cross-sectional area.
- The **resistivity in terms of the average collision time** τ is $\rho = \frac{m_e}{ne^2\tau}$.
- The **resistance in terms of resistivity** is $R = \rho\frac{\ell}{A}$. The **SI unit of resistance** is 1 ohm = 1 Ω = 1 V/A.
- The handy dandy **Ohm's Law** is defined as $V = IR$.
- The **current density** is defined as $j = \frac{I}{A}$.
- This implies that we can write **Ohm's law in terms of the current density**, $j = \frac{E}{\rho}$.
- The **change of resistance with temperature** is $\delta R = \alpha R_0 \delta T$, where α is the temperature coefficient of resistivity. You can find this in a table somewhere; google it.
- Resistors are things one can also put on a wire! Woo, circuits!
  - For resistors connected in series, the same current flows through each, and the net potential difference is the sum of the individual potential differences. So the **series combination of resistors** is additive, i.e. $R = R_1 + R_2 + R_3 + ...$
  - For resistors connected in parallel, the total current is the sum of the individual parallel currents, and the potential difference across each is the same. So the **parallel combination of resistors** is $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + ...$

4 Magnetostatics

4.1 Magnetic Force and Field

- The **magnetic force exerted on moving charge by current in long wire** where the $-$ sign means $\mathbf{F}$ is attractive for $\mathbf{v}$ parallel to I and $+$ means $\mathbf{F}$ is parallel to I for $\mathbf{v}$ radially outward. For $\mathbf{v}$ tangent to circles around a wire, $F = 0$. So we can say that,

$$F = \begin{cases} \frac{\mu_0}{2\pi}\frac{qvI}{r} \\ 0 \end{cases}$$

- The **permeability constant** is defined as $\mu_0 = 4\pi \times 10^{-7}\frac{\mathrm{Ns}^2}{\mathrm{C}^2} \approx 1.26 \times 10^{-6}\frac{\mathrm{Ns}^2}{\mathrm{C}^2}$.

- The **force exerted by a magnetic field on a moving charge** is defined as $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$. Of course, we can describe the magnitude of this vector force, as it is a cross product, it yields a sine: $F = qvB\sin\theta$.

- Turns out that the law of superposition can help lead us to a more profound form of this definition for magnetic force. What if there are coupled magnetic and electric fields? Well we have what is called the **Lorentz Force**,

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right).$$

  The **superposition principle** also apply to magnetic fields and forces as well. The magnetic forces and magnetic fields produced by different currents combine via vector addition.

- The **Biot-Savart Law** is defined as,

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{Id\mathbf{s} \times \mathbf{r}}{r^3}$$

  We can integrate both sides using the generalized magnitudes,

$$B = \int dB \quad = \quad \int \frac{\mu_0}{4\pi} \frac{I ds \sin\theta}{r^2}$$

Now let's try to define what I just wrote for you. The above equation illustrates the contribution to the magnetic field where **r** is the vector from the current element $Id\mathbf{s}$ to some point P.

- Wow, Biot-Savart is really ugly. If only there was something easier to use. Well, there is, if your object has really nice symmetry. If it does, you'll want to use **Ampére's Law**, which states,

  $$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{encl.}.$$

  Like how we did for Gauss's law and the electric field, we can make things easier for us. We can get rid of the dot product as when we have nice symmetry the **B** and the $d\mathbf{l}$ are parallel. Thus,

  $$\oint B_{||} d; \quad = \quad \begin{cases} B(2\pi 4) & \text{(current flowing along a line)} \\ B(Bl) & \text{(current flowing around a cylinder)} \end{cases}.$$

- The **magnetic field of current in a long wire** (i.e. infinite), which can be defined by the help of the right hand rule or if you're feeling more mathy, Ampére's law. It is $B = \frac{\mu_0 I}{2\pi r}$.
- Speaking of magnetic field, we should probably talk a bit about it's units as it is something we can empirically identify. The **SI unit of the magnetic field** is the Tesla (after the lovely Nikola Tesla, of course), so 1 tesla = 1 T = $1 \frac{\text{N}}{\text{C m/s}}$.

References

[1] Physics, for Engineers and Scientists. Third Ed. Vol. 2. 2007. Hans C. Ohanian. John T. Market.

[2] Introduction to Electrodynamics. 3rd/4th Ed. Prentice Hall. David J. Griffiths.