Let $x = A\sin(\omega t+\phi)$

$$\frac{dx}{dt} = A\omega\cos(\omega t+\phi)$$

$$\frac{d^2x}{dt^2} = -A\omega^2\sin(\omega t+\phi)$$

substitute into differential eqn —

$$-A\omega^2\sin(\omega t+\phi) + \frac{k}{M}A\sin(\omega t+\phi) = 0$$

True if $\omega^2 = k/M$!

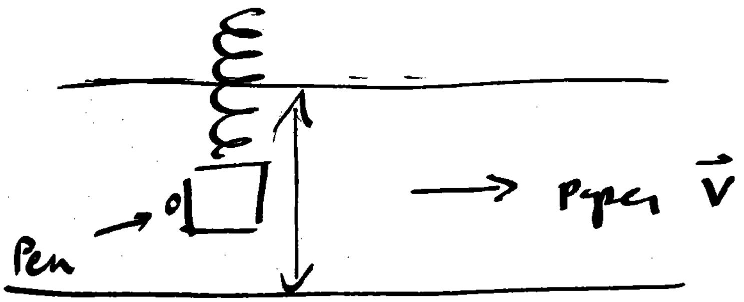
So our little differential eqn has solns

either $x(t) = A\cos(\omega t+\phi)$ OR $x(t) = A\sin(\omega t+\phi)$

Does it make sense?

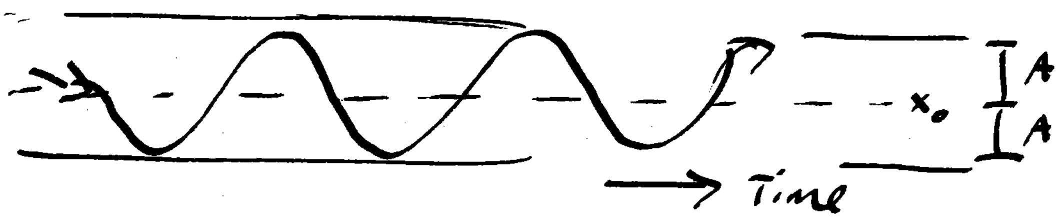
Harmonic Functions

⇒ why we call it Simple Harmonic Motion!



let mass on spring oscillate on frictionless horizontal plane

View from above



$F = -kx \qquad x_0 = 0$

$ma = -kx$

$m\dfrac{d^2x}{dt^2} = -kx$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

2ND order ordinary differential equation

differential equation
equation of Motion for
a Simple Harmonic
oscillator

Solve this for $X(t)$ → tells where spring is
as a fn of time
This is what you need usually

Let $x = A\cos(\omega t + \phi)$

Amplitude — Frequency — initial phase — all just constants for Now

general soln ... constants set by "initial" and "boundary" conditions Specific to problem

$$\frac{dx}{dt} = -A\omega\sin(\omega t + \phi)$$

$$\frac{d^2x}{dt^2} = -A\omega^2\cos(\omega t + \phi)$$

Substitute into differential eqn

$$-A\omega^2\cos(\omega t + \phi) + \frac{k}{M}A\cos(\omega t + \phi) = 0$$

True <u>if</u> $\omega^2 = k/M$ or $\omega = \pm\sqrt{k/M}$

The motion is Periodic

repeats in Time T, called the Period

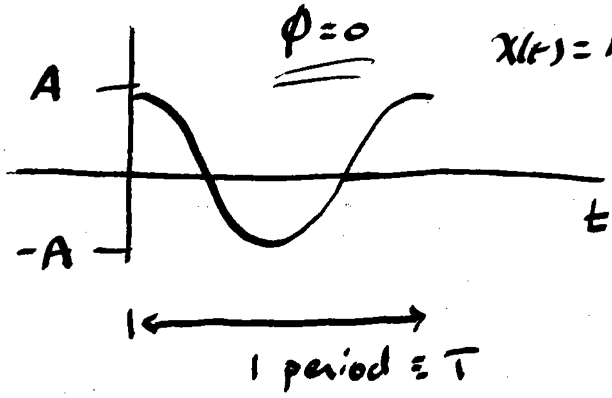
Frequency $\equiv \frac{1}{T}$ units of $\frac{1}{s}$ or Hertz, Hz

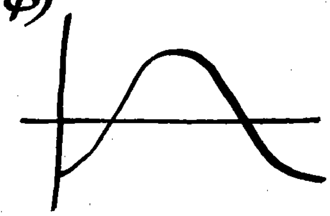
$X(t) = A\cos(\omega t + \phi)$ ← initial phase angle

↑ Amplitude of motion

frequency $\omega = \frac{2\pi}{T}$ in radians

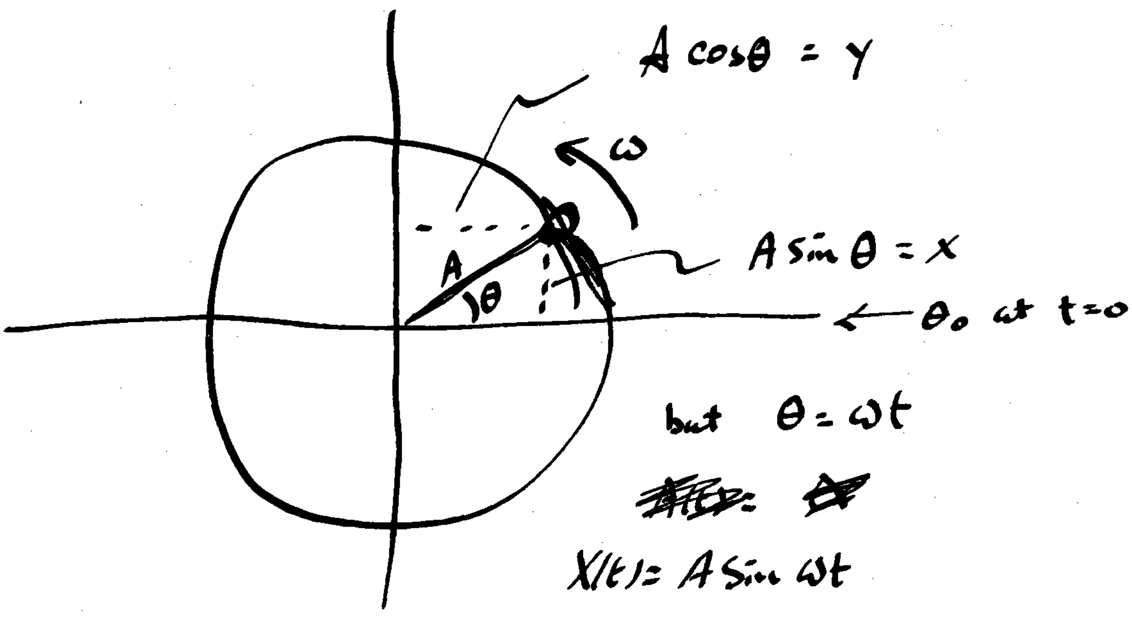
$\phi = 0$ $X(t) = A\cos(\omega t+\phi)$





Curve if $\phi = \pi$

Circular motion can be thought of as a Superposition of linear SHM in 2 dimensions



but $\theta = \omega t$

~~X~~: ~~X~~

$X(t) = A\sin\omega t$

$Y(t) = A\cos\omega t$

[illegible] phase angle into argument

Spring

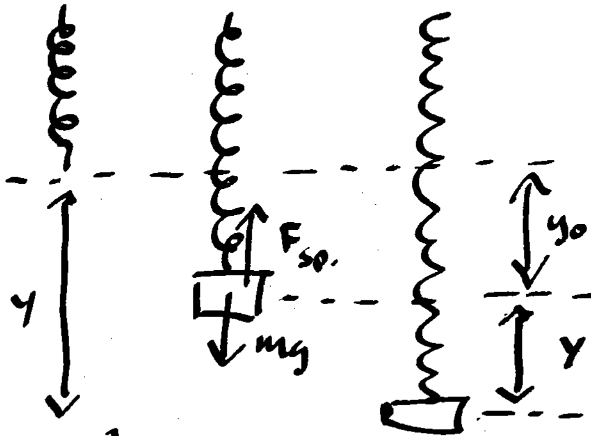
AT Maximum Amplitude ... know all E is PE

$$E = \tfrac{1}{2} k x^2 = \tfrac{1}{2} k A^2$$

$\therefore$ $\boxed{\text{Total Energy in a SHO is } \tfrac{1}{2} k A^2}$

Example

Show a mass oscillating on a vertical spring executes simple Harmonic Motion



equil. length (actual) of spring w/ no mass

length w/ mass in equil.

oscillate about $y = y^0$

$$\Sigma F_y = 0 \Rightarrow k y_0 = mg \qquad y_0 = mg/k$$

$$\Sigma F_y = -ky + mg = ma = m\frac{d^2y}{dt^2}$$

$\cancel{y' = y - y_0}$

$\cancel{y = y_0 + y'}$ ~~[illegible]~~

Let $y \rightarrow y' + \frac{mg}{k}$

$$\frac{dy}{dt} = \frac{dy'}{dt} \quad \text{and} \quad \frac{d^2y}{dt^2} = \frac{d^2y'}{dt^2}$$

So

ΣF_y becomes $\boxed{-ky' = m\frac{d^2y'}{dt^2}}$

This is eqn of Motion for SHO we know how to solve!

Tells us ~~spring~~ mass oscillates w/ SHM about y_0

$F = -k y'$ where $y' = y - \frac{mg}{k}$

$y'(t) = A\cos(\omega t + \theta_0)$

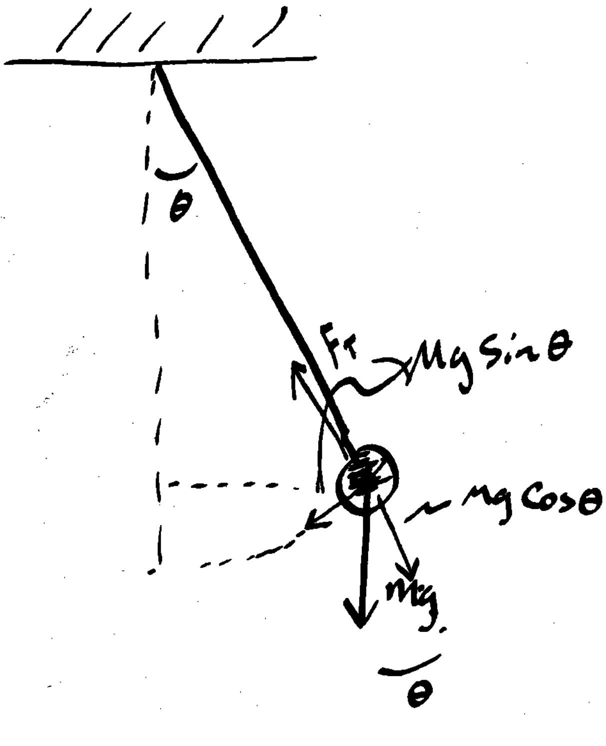
w/ $\omega = \sqrt{\frac{k}{m}}$

Doesn't Δ frequency

gravity shifts the equilibrium point of motion from $y = 0$ to $y' = 0$

All other results are the same as for SHM in variable y' !

<u>example</u> Simple Pendulum



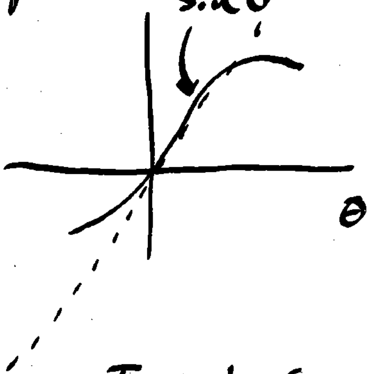
restoring force $F = mg\sin\theta$

<u>But</u> $\sin\theta \approx \theta$ for small θ

→ Whoa! Where did I get this magic ?!

Digress a little

$\sin\theta \sim f(x) = \theta$



Look at small θ

$\sin\theta \approx \theta$

Taylor's Series expansion about $x = a$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \dots + \frac{f^{(n-1)}(a)(x-a)^{n-1}}{(n-1)!} + \dots$$

Look at

$f(x) = \sin\theta$ about $\theta = 0$

$$\sin\theta = \sin(0) + \cos(0)(\theta-0) + \frac{[-\sin(0)](\theta-0)^2}{2!} + \frac{[-\cos(0)](\theta-0)^3}{3!} + \dots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \dots$$

for small θ

$\sin\theta \approx \theta$

Error in our Approximation

θ_{deg}	θ_{rad}	$\sin\theta$	$\frac{\theta}{\sin\theta}$
1.0	0.017453	0.017452	.006%
10	0.1745	0.1736	0.51%
30			4.7%

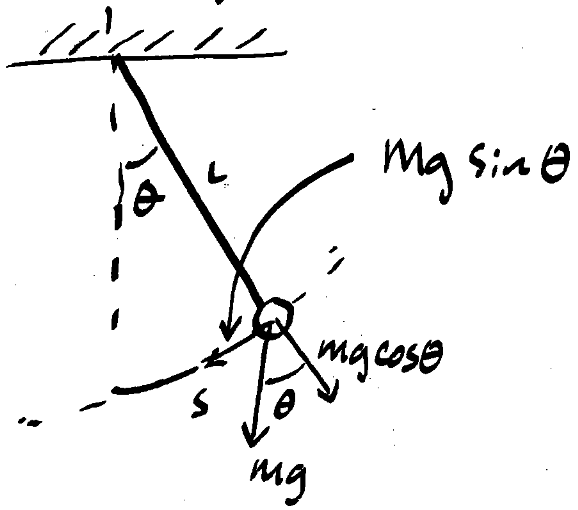
Taylor's Series expansion of $\cos x$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$\cos x \approx 1$ for x small

This is a Very useful Tool in Physics !!

back To Simple Pendulum



$$s = L\theta$$

$$m\frac{d^2s}{dt^2} = -mg\sin\theta \approx -mg\theta = -mg\,s/L$$

$$\cancel{m\frac{d^2(L\theta)}{dt^2} = -mg\theta}$$

$$\boxed{\frac{d^2s}{dt^2} + \frac{g}{L}s = 0}$$

$$\cancel{\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta}$$

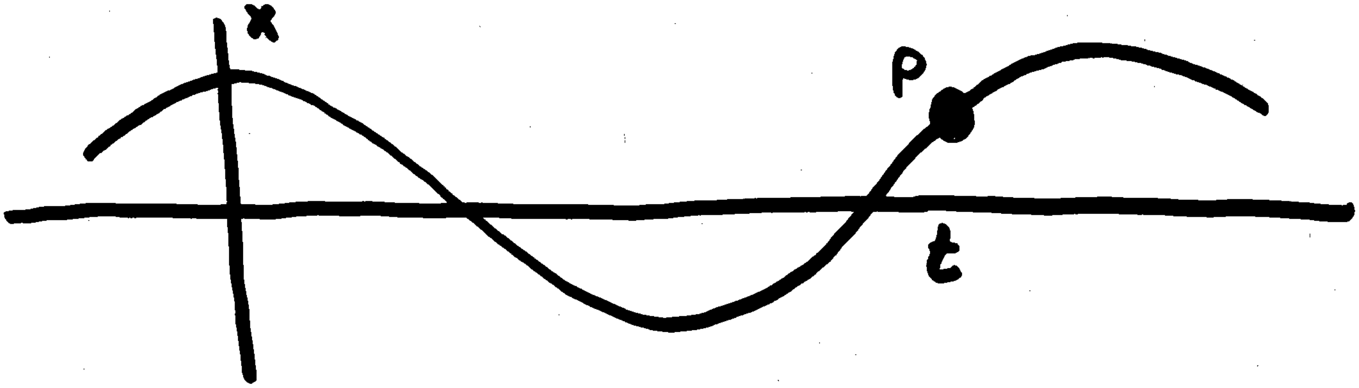
Same form as SHO differential eqn

$\therefore$

$$s(t) = A\cos(\omega t + \phi) \quad \text{where}$$

$$\omega = \sqrt{g/L}$$

Mass attached to spring oscillates Back + Forth as shown in position-Time graph below



AT Point P the mass has

1. Positive Velocity, positive Acceleration
2. Positive velocity, Negative Accel.
3. Negative velocity, Positive Accel.
4. Negative velocity, Negative Accel.
5. Zero velocity but acceleration
6. Some velocity but zero Accel.
7. Zero velocity, zero accel.

A simple pendulum has a period T on Earth. How does this compare to the period of the same pendulum if taken to the moon?

1. $T_{Earth} > T_{Moon}$
2. $T_{Earth} < T_{Moon}$
3. $T_{Earth} = T_{Moon}$

$$\omega = \sqrt{g/L}$$

$$\omega = \frac{2\pi}{T}$$

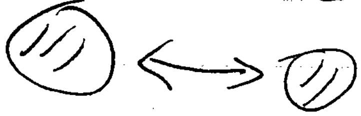
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$

$$g_{Moon} < g_{Earth}$$

$$\therefore \quad T_{Moon} > T_{Earth} \Rightarrow 2$$

Theoretical Chemistry Example

quantized
Vibrational energy levels



usually emit/absorb in IR

Diatomic Molecule
or
Atomic Bond

Given $F = -\frac{C}{r^2} + \frac{D}{r^3}$

What is Frequency and period of vibration?

What is equilibrium position $\equiv r_0$

at $r_0 = r$ $\quad F = 0$

$\therefore$

$$\frac{C}{r_0^{\,2}} = \frac{D}{r_0^{\,3}} \Rightarrow r_0 = D/C$$

Consider small oscillations about r_0

$$r = r_0 + x$$

$$F = -\frac{C}{(r_0+x)^2} + \frac{D}{(r_0+x)^3}$$

recall from Calculus $\rightarrow$ Taylor's series expansions

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \ldots + \frac{f^{(n-1)}(a)(x-a)^{n-1}}{(n-1)!}$$

Thus

$$(1+\alpha)^{-2} = 1 - 2\alpha + 3\alpha^2 + \dots$$

$$(1+\alpha)^{-3} = 1 - 3x + 6x^2 + \dots$$

$$F = -\frac{C}{(r_0+x)^2} + \frac{D}{(r_0+x)^3} = -\frac{C}{r_0^2\left(1+\frac{x}{r_0}\right)^2} + \frac{D}{r_0^3\left(1+\frac{x}{r_0}\right)^3}$$

$$F = -\frac{C}{r_0^2}\Big(1+\underbrace{\tfrac{x}{r_0}}_{\equiv\alpha}\Big)^{-2} + \frac{D}{r_0^3}\Big(1+\underbrace{\tfrac{x}{r_0}}_{\equiv\alpha}\Big)^{-3}$$

α is small

sub in Taylor's Series expansion, Drop terms $O\left(\frac{x}{r_0}\right)^2$ or smaller

$$F = -\frac{C}{r_0^2}\left(1 - 2\frac{x}{r_0}\right) + \frac{D}{r_0^3}\left(1 - 3\frac{x}{r_0}\right)$$

$$F = \underbrace{-\frac{C}{r_0^2} + \frac{D}{r_0^3}}_{0} + C2\frac{x}{r_0} - \frac{D3}{r_0}x$$

0 by def of equilibrium position r_0 as shown earlier

$$m\frac{d^2x}{dt^2} = -\frac{1}{r_0}(3D+2C)x$$

constant $\equiv K$

$$\frac{d^2x}{dt^2} + \frac{K}{m}x = 0$$

SHM w/ $\omega = \sqrt{\frac{K}{m}}$

$$\text{Period} = \frac{2\pi}{\omega}$$