# Physics 123 - February 20, 2013

Last Time

## Fourier Analysis

$m \frac{2\pi}{\lambda} x$

$m^{\underline{th}}$ term of harmonic series is a harmonic w/ $\lambda \to \frac{\lambda}{m}$
"Integral submultiple"

$$f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos(mkx) + \sum_{m=1}^{\infty} B_m \sin(mkx)$$

Some periodic function w/ spatial period $\lambda$

$A_0$, $A_m$, $B_m$ are constants

$\infty$ # of these in principle

As m increases the harmonic term has smaller $\lambda$

Finding $A_0, A_m, B_m$ is Fourier analysis

Find $A_0$

$$\int_0^\lambda f(x)\,dx = \int_0^\lambda \frac{A_0}{2}\,dx + \int_0^\lambda \left[\sum A_m \cos mkx + \sum B_m \sin mkx\right] dx$$

$$\int_0^\lambda f(x)\,dx = \frac{A_0}{2}\lambda \longrightarrow \boxed{A_0 = \frac{2}{\lambda}\int_0^\lambda f(x)\,dx}$$

To find $A_m, B_m$, use:

$$\int_0^\lambda \sin(akx)\cos(bkx)\,dx = 0$$

$$\int_0^\lambda \sin(akx)\sin(bkx)\,dx = \frac{\lambda}{2}\delta_{ab}$$

$= 1$ when $a=b$
$= 0$ when $a \neq b$

$$\int_0^\lambda \cos(akx)\cos(bkx)\,dx = \frac{\lambda}{2}\delta_{ab}$$

$$f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos(mkx) + \sum_{m=1}^{\infty} B_m \sin(mkx)$$

to find $A_\ell$ $\longrightarrow$ mult both sides by $\cos(\ell kx)$ and integrate from $0 \to \lambda$

$$\int_0^\lambda f(x)\cos(\ell kx)\,dx = \frac{A_0}{2}\cancel{\int_0^\lambda \cos(\ell kx)\,dx}^{\,0} + \sum_{m=1}^{\infty} A_m \overbrace{\int_0^\lambda \cos(mkx)\cos(\ell kx)\,dx}^{\text{all 0 except for when } m=\ell}$$

$$+ \sum_{m=1}^{\infty} B_m \cancel{\int_0^\lambda \sin(mkx)\cos(\ell kx)\,dx}^{\,0}$$

$$\int_0^\lambda f(x)\cos(\ell kx)\,dx = A_\ell \underbrace{\int_0^\lambda \cos^2(\ell kx)\,dx}_{=\frac{\lambda}{2}}$$

$$A_\ell = \frac{2}{\lambda}\int_0^\lambda f(x)\cos(\ell kx)\,dx$$

To find $B_\ell$, mult. by $\sin(\ell k x)$ and integrate over $x$ from $0$ to $\lambda$

$$\boxed{B_\ell = \frac{2}{\lambda}\int_0^\lambda f(x)\sin(\ell k x)\,dx}$$

## Example

Suppose we have function $F(t)$ periodic in time w/ period $T$

$$F(t+T) = F(t)$$

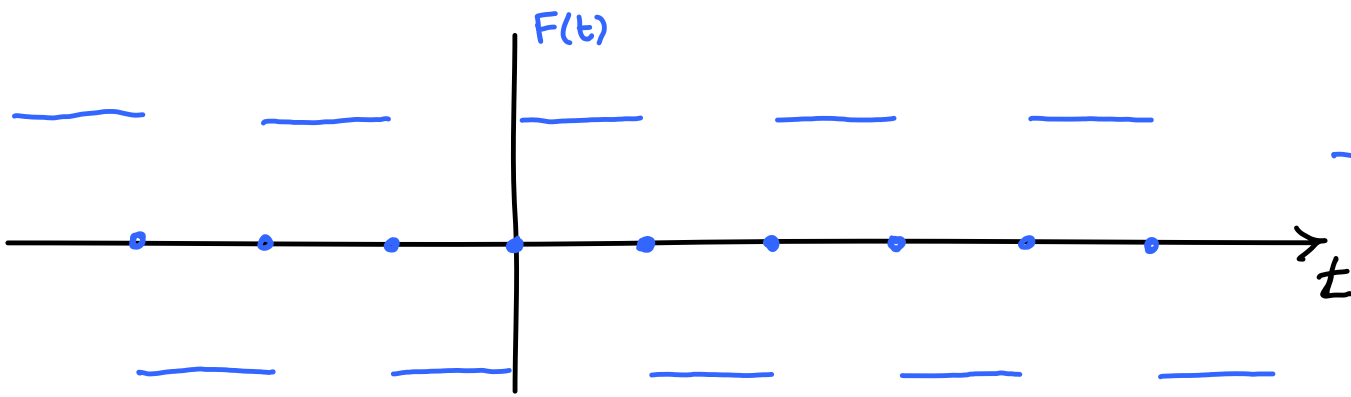
Such that

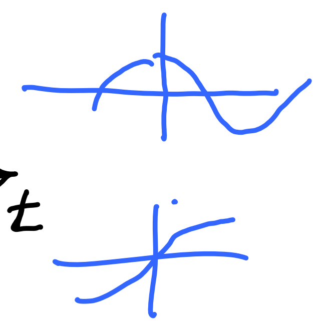
$$F(t) = +1 \quad \text{for} \quad 0 < t < \frac{T}{2}$$

$$F(t) = -1 \quad \text{for} \quad T/2 < T < T$$

$$F(t) = 0 \quad \text{for} \quad t = 0, \frac{T}{2}, T, \dots$$

Find a solution for $F(t)$ using Fourier Analysis.



$$F(t) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos(m\omega t) + \sum_{m=1}^{\infty} B_m \sin(m\omega t)$$

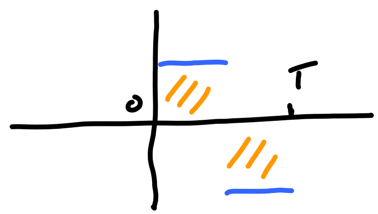
Even

$F(x) = F(-x)$

$F(x) = -F(-x)$ odd

$F(t)$ is an odd function $\longrightarrow$ all $A_m = 0$

$$A_0 = \frac{2}{T}\int_0^T F(t)\,dt$$



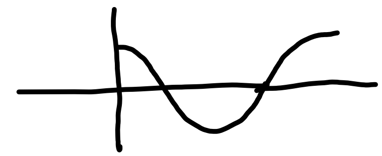

$$\int_0^T F(t)\,dt = 0 \longrightarrow A_0 = 0$$

Find $B_m$

$$\int_0^T F(t)\sin(l\omega t) = B_l \frac{T}{2}$$

$$B_l = \frac{2}{T}\left\{\int_0^{T/2} \sin(l\omega t)\,dt - \int_{T/2}^{T} \sin(l\omega t)\,dt\right\}$$

$$B_\ell = \frac{2}{T}\left\{ -\frac{1}{\ell\omega}\cos(\ell\omega t)\Big|_{0}^{T/2} + \frac{1}{\ell\omega}\cos(\ell\omega t)\Big|_{T/2}^{T} \right\}$$



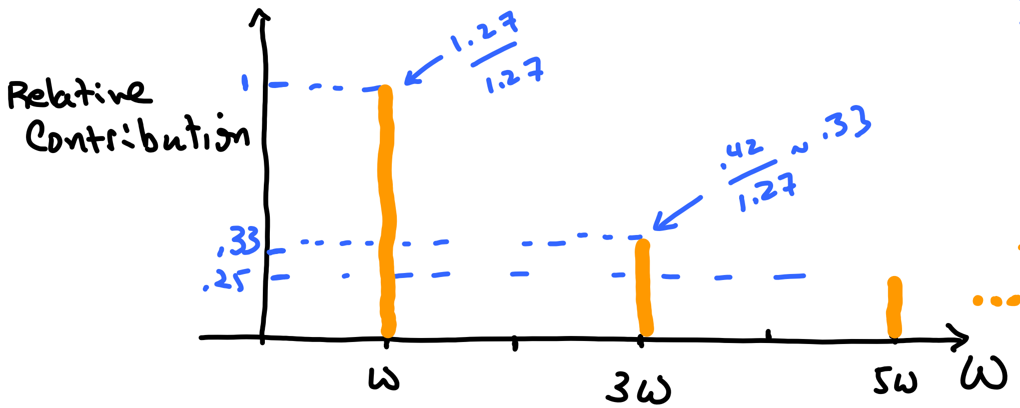
$$B_\ell = \underbrace{\frac{2}{T}\,\frac{1}{\ell\omega}}_{\frac{1}{\ell\pi} = \frac{2\,T}{T\,\ell\,2\pi}} \left\{ \underbrace{-\cos\left(\ell\frac{2\pi}{T}\frac{T}{2}\right)}_{\cos(\ell\pi)} + 1 + \underbrace{\cos\left(\ell\frac{2\pi}{T}T\right)}_{\substack{\cos(\ell 2\pi) \\ =1 \text{ for} \\ \text{integer } \ell}} - \underbrace{\cos\left(\ell\frac{2\pi}{T}\frac{T}{2}\right)}_{\cos(\ell\pi)} \right\}$$

$$B_\ell = \frac{2}{\ell\pi}\left[1 - \cos(\ell\pi)\right]$$

$$B_1 = \frac{2}{\pi}\left[1 - \cos(\pi)\right] = \frac{4}{\pi} \sim 1.27, \quad B_2 = 0, \quad B_3 = \frac{4}{3\pi} \sim 0.42, \quad \cdots$$

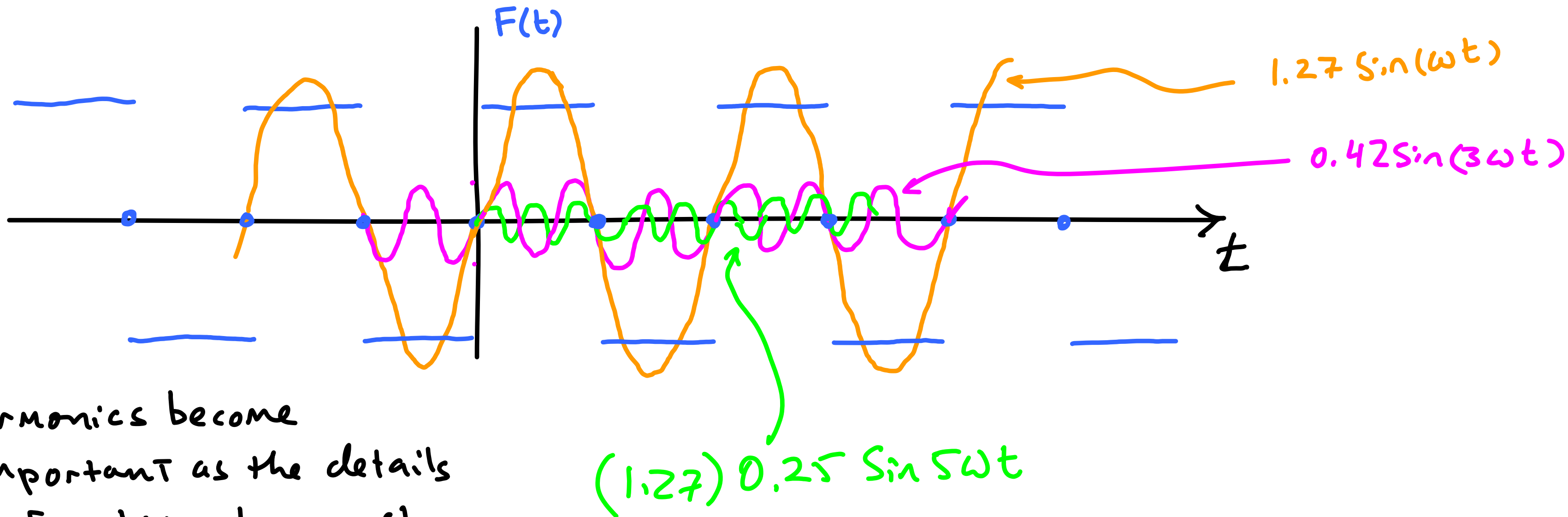
$$F(t) = \frac{4}{\pi}\sin(\omega t) + \frac{4}{3}\pi\sin(3\omega t) + \frac{4}{5\pi}\sin(5\omega t) + \cdots$$

$\omega$ is the Fundamental frequency for $F(t)$



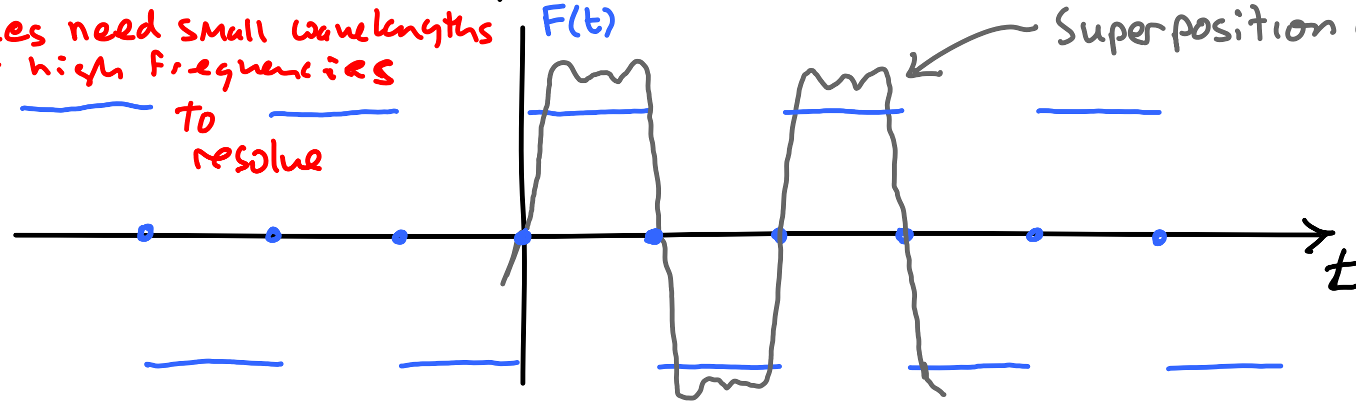

$3\omega, 5\omega$ are harmonics of the fundamental

Description of $F(t)$ in "Frequency Space" "$\omega$-Space"



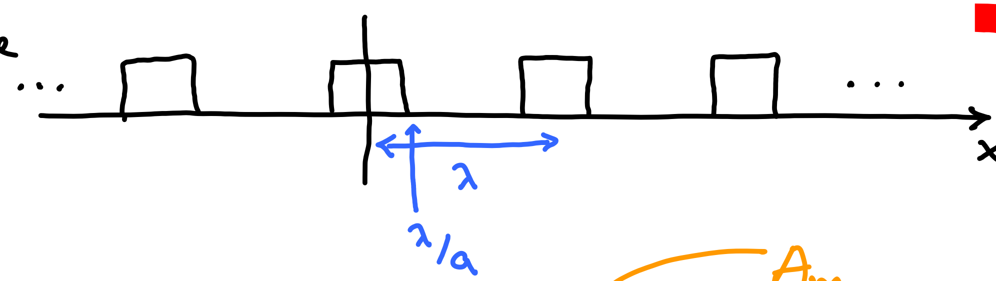

Higher harmonics become more important as the details in the Function become sharp

Small Features need small wavelengths or high frequencies to resolve




Adding successive Terms brings shape closer to F(t)

Suppose we have a square wave




Fourier Series
(given w/ out proof here)
Description of this is

$$F(x) = \frac{2}{a} + \sum_{m=1}^{\infty} \overbrace{\boxed{\frac{4}{a}\left[\frac{\sin\left(\frac{m2\pi}{a}\right)}{\frac{m2\pi}{a}}\right]}}^{A_m} \cos(m\underline{k}x)$$

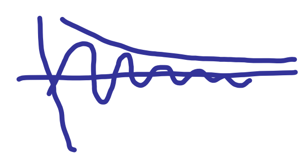
$k \to$ spatial frequency

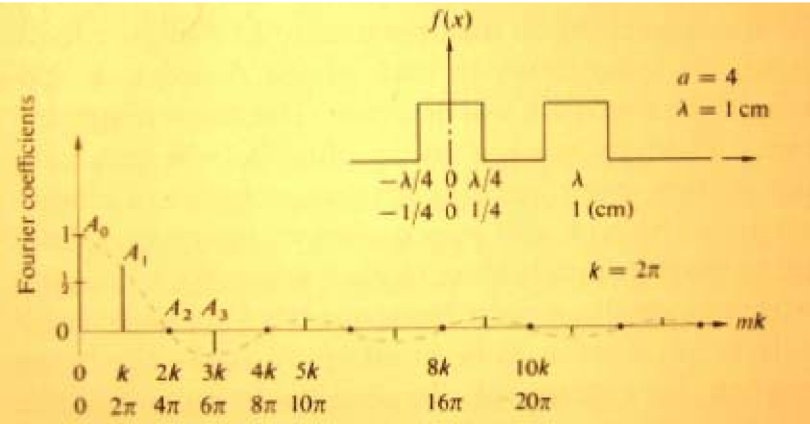
$$\frac{2}{m\pi}\sin\left(\frac{m2\pi}{a}\right)$$

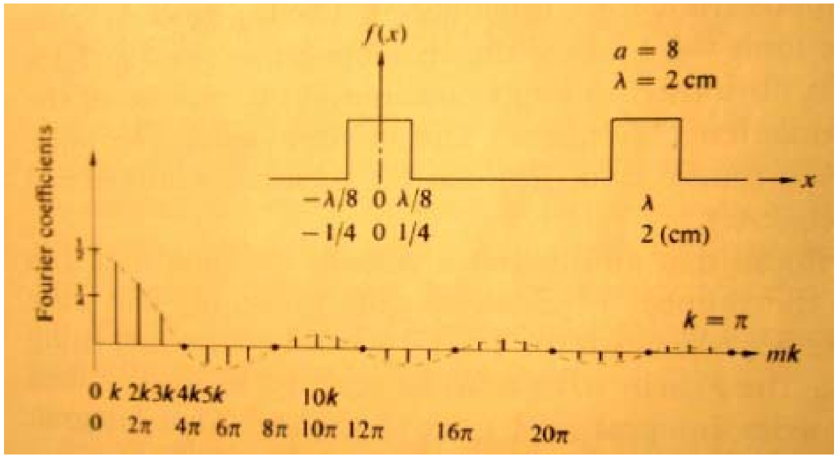
$$\frac{2\pi}{\lambda}$$

Coefficients go as $\sin\left(\frac{2\pi}{a/m}\right)$ w/ Sin Amplitude

damped as $\sim \frac{1}{m}$

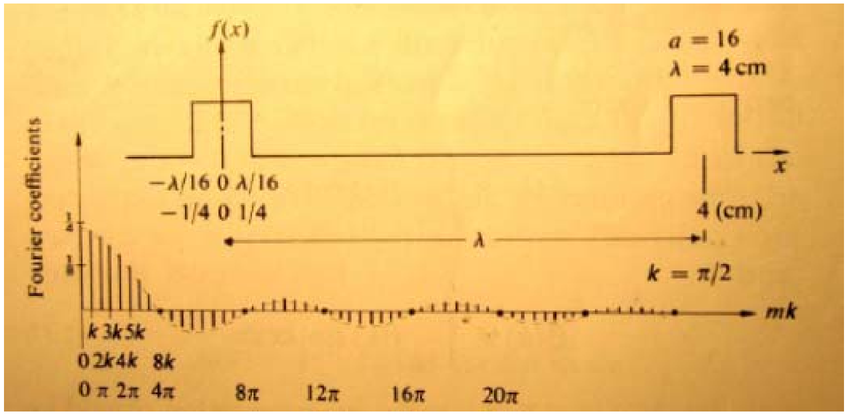





Optics

Hecht

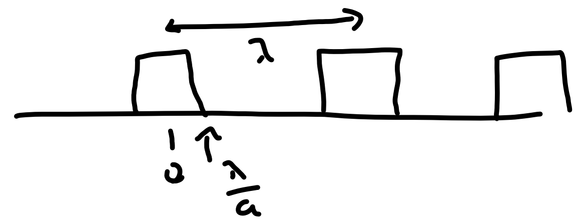


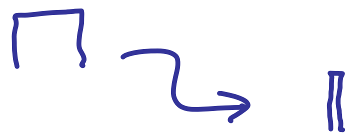

Keep square size the same and increase $\lambda$

→ frequency spectrum becomes denser

→ higher frequencies become more important (relatively larger coefficients)

$\sin\left(\frac{m\,2\pi}{a}\right)$




as a gets large  Square becomes narrow in x

- and -

Sin that governs the spatial frequency envelope gets broad

Fourier series

$$F(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos(mkx) + \sum_{m=1}^{\infty} B_m \sin mkx$$

in limit $\lambda \to \infty$

$$F(x) = \frac{1}{\pi}\left[\int_0^{\infty} A(k)\cos(kx)\,dx + \int_0^{\infty} B(k)\sin(kx)\,dx\right]$$

Provided

Cosine

> TRANSFORMS of $F(x)$

Sine

$$A(k) = \int_{-\infty}^{\infty} F(x)\cos(kx)\,dx$$

and

$$B(k) = \int_{-\infty}^{\infty} F(x)\sin(kx)\,dx$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\mathrm{Re}(e^{i\theta}) = \cos\theta$$

$$A(k) = \int_{-\infty}^{\infty} F(x)\cos(kx)\,dx$$

$$= \int_{-\infty}^{\infty} F(x)\,\mathrm{Re}(e^{ikx})\,dx$$

$$A(k) = \int_{-\infty}^{\infty} F(x)\,e^{ikx}\,dx$$

$$\lambda = h/p \qquad p \sim \frac{1}{\lambda} \sim k$$