Physics 123 - Spring 2013 - solutions to problem set 11

1.
39-41

(*a*) The probability of the electron passing through the barrier is given by Eqs. 38-17a and 38-17b.

$$T = e^{-2\ell\frac{\sqrt{2m(U_0-E)}}{\hbar}}$$

$$2\ell\frac{\sqrt{2m(U_0-E)}}{\hbar} = 2(0.25\times10^{-9}\,\text{m})\frac{\sqrt{2(9.11\times10^{-31}\,\text{kg})(1.2\,\text{eV})(1.60\times10^{-19}\,\text{J/eV})}}{(1.055\times10^{-34}\,\text{J}\cdot\text{s})} = 2.803$$

$$T = e^{-2.803} = 6.063\times10^{-2} \approx \boxed{6.1\%}$$

(*b*) The probability of reflecting is the probability of NOT tunneling, and so is $\boxed{93.9\%}$.

2.
39-3

The value of ℓ ranges from 0 to $n-1$. Thus for $n=3$, $\ell=0,1,2$. For each ℓ the value of m_ℓ can range from $-\ell$ to $+\ell$, or $2\ell+1$ values. For each m_ℓ there are 2 values of m_s. Thus the number of states for each ℓ is $2(2\ell+1)$. The number of states is $N = 2(0+1)+2(2+1)+2(4+1)$ $= \boxed{18\text{states}}$. We start with $\ell=0$, and list the quantum numbers in the order (n,ℓ,m_ℓ,m_s).

$(3,0,0,-\frac{1}{2}), (3,0,0,+\frac{1}{2}), (3,1,-1,-\frac{1}{2}), (3,1,-1,+\frac{1}{2}), (3,1,0,-\frac{1}{2}), (3,1,0,+\frac{1}{2}),$
$(3,1,1,-\frac{1}{2}), (3,1,1,+\frac{1}{2}), (3,2,-2,-\frac{1}{2}), (3,2,-2,+\frac{1}{2}), (3,2,-1,-\frac{1}{2}), (3,2,-1,+\frac{1}{2}),$
$(3,2,0,-\frac{1}{2}), (3,2,0,+\frac{1}{2}), (3,2,1,-\frac{1}{2}), (3,2,1,+\frac{1}{2}), (3,2,2,-\frac{1}{2}), (3,2,2,+\frac{1}{2})$

3.
39-9

For a given n, $0 \le \ell \le n-1$. Since for each ℓ the number of possible states is $2(2\ell+1)$, the number of possible states for a given n is as follows.

$$\sum_{\ell=0}^{n-1} 2(2\ell+1) = 4\sum_{\ell=0}^{n-1}\ell + \sum_{\ell=0}^{n-1} 2 = 4\left(\frac{n(n-1)}{2}\right) + 2n = \boxed{2n^2}$$

4.
39-12

To show that the ground-state wave function is normalized, we integrate $|\psi_{100}|^2$ over all space. Use substitution of variables and an integral from Appendix B-5.

$$\int_{\text{all space}} |\psi_{100}|^2 \, dV = \int_0^\infty \frac{1}{\pi r_0^3} e^{-\frac{2r}{r_0}} 4\pi r^2 dr \,; \quad \text{let } x = \frac{2r}{r_0} \;\rightarrow\; r = \tfrac{1}{2} r_0 x \,, \; dr = \tfrac{1}{2} r_0 dx$$

Note that if $r = 0, x = 0$ and if $r = \infty, x = \infty$.

$$\int_{\text{all space}} |\psi_{100}|^2 \, dV = \int_0^\infty \frac{1}{\pi r_0^3} e^{-\frac{2r}{r_0}} 4\pi r^2 dr = \frac{4\pi}{\pi r_0^3} \int_0^\infty e^{-x} \left(\tfrac{1}{4} r_0^2 x^2\right)\left(\tfrac{1}{2} r_0 dx\right) = \tfrac{1}{2} \int_0^\infty e^{-x} x^2 dx = \tfrac{1}{2}(2!) = 1$$

And so we see that the ground-state wave function is normalized.

5.
39-14

The state $n = 2$, $\ell = 0$ must have $m_\ell = 0$ and so the wave function is $\psi_{200} = \frac{1}{\sqrt{32\pi r_0^3}}\left(2 - \frac{r}{r_0}\right)e^{-\frac{r}{2r_0}}$.

(a) $(\psi_{200})_{r=4r_0} = \frac{1}{\sqrt{32\pi r_0^3}}\left(2 - \frac{4r_0}{r_0}\right)e^{-\frac{4r_0}{2r_0}} = \boxed{-\frac{1}{\sqrt{8\pi r_0^3}}e^{-2}}$

(b) $\left(|\psi_{200}|^2\right)_{r=4r_0} = \frac{1}{32\pi r_0^3}\left(2 - \frac{4r_0}{r_0}\right)^2 e^{-2\frac{4r_0}{2r_0}} = \boxed{\frac{1}{8\pi r_0^3}e^{-4}}$

(c) $P_r = \left(4\pi r^2 |\psi_{200}|^2\right)_{r=4r_0} = 4\pi(4r_0)^2\left(\frac{1}{8\pi r_0^3}e^{-4}\right) = \boxed{\frac{8}{r_0}e^{-4}}$

6.
39-16

(a) To find the probability, integrate the radial probability distribution for the ground state. follow Example 39-4, and use the last integral in Appendix B-4.

$$P = \int_0^{r_0} 4\frac{r^2}{r_0^3}e^{-\frac{2r}{r_0}}dr \;;\; \text{let } x = 2\frac{r}{r_0} \;\rightarrow$$

$$P = \tfrac{1}{2}\int_0^2 x^2 e^{-x}dx = \tfrac{1}{2}\left[-e^{-x}\left(x^2 + 2x + 2\right)\right]_0^2 = 1 - 5e^{-2} = 0.32 = \boxed{32\%}$$

(b) We follow the same process here.

$$P = \int_{r_0}^{2r_0} 4\frac{r^2}{r_0^3}e^{-\frac{2r}{r_0}}dr \;;\; \text{let } x = 2\frac{r}{r_0} \;\rightarrow$$

$$P = \tfrac{1}{2}\int_2^4 x^2 e^{-x}dx = \tfrac{1}{2}\left[-e^{-x}\left(x^2 + 2x + 2\right)\right]_2^4 = 5e^{-2} - 13e^{-4} = 0.44 = \boxed{44\%}$$

7.
39-29

(*a*) For carbon, $Z = 6$. We start with the $n = 1$ shell, and list the quantum numbers in the order (n, ℓ, m_ℓ, m_s).

$$\boxed{(1,0,0,-\tfrac{1}{2}),(1,0,0,+\tfrac{1}{2}),(2,0,0,-\tfrac{1}{2}),(2,0,0,+\tfrac{1}{2}),(2,1,-1,-\tfrac{1}{2}),(2,1,-1,+\tfrac{1}{2})}$$

Note that, without additional information, there are other possibilities for the last two electrons.

(*b*) For aluminum, $Z = 13$. We start with the $n = 1$ shell, and list the quantum numbers in the order (n, ℓ, m_ℓ, m_s).

$$\boxed{\begin{array}{l}(1,0,0,-\tfrac{1}{2}),(1,0,0,+\tfrac{1}{2}),(2,0,0,-\tfrac{1}{2}),(2,0,0,+\tfrac{1}{2}),(2,1,-1,-\tfrac{1}{2}),(2,1,-1,+\tfrac{1}{2}),\\ (2,1,0,-\tfrac{1}{2}),(2,1,0,+\tfrac{1}{2}),(2,1,1,-\tfrac{1}{2}),(2,1,1,+\tfrac{1}{2}),(3,0,0,-\tfrac{1}{2}),(3,0,0,+\tfrac{1}{2}),(3,1,-1,-\tfrac{1}{2})\end{array}}$$

Note that, without additional information, there are other possibilities for the last electron.

8.
39-31

Since the electron is in its lowest energy state, we must have the lowest possible value of n. Since $m_\ell = 2$, the smallest possible value of ℓ is $\boxed{\ell = 2}$, and the smallest possible value of n is $\boxed{n = 3}$.

9.
39-32

Limiting the number of electron shells to six would mean that the periodic table stops with radon (Rn), since the next element, francium (Fr), begins filling the seventh shell. Including all elements up through radon means $\boxed{86}$ elements.

10.
39-34

The third electron in lithium is in the 2*s* subshell, which is outside the more tightly bound filled 1*s* shell. This makes it appear as if there is a "nucleus" with a net charge of +1e. Thus we use the energy of the hydrogen atom.

$$E_2 = -\frac{(13.6\text{eV})}{n^2} = -\frac{(13.6\text{eV})}{2^2} = -3.4\text{eV}$$

We predict the binding energy to be $\boxed{3.4\text{ eV}}$. Our assumption of complete shielding of the nucleus by the 2*s* electrons is probably not correct. The partial shielding means the net charge of the "nucleus" is higher than +1e, and so it holds the outer electron more tightly, requiring more energy to remove it.

11.
39-36

The energy levels of the infinite square well are given in Eq. 38-13. Each energy level can have a maximum of two electrons, since the only quantum numbers are n and m_s. Thus the lowest energy level will have two electrons in the $n = 1$ state, two electrons in the $n = 2$ state, and 1 electron in the $n = 3$ state.

$$E = 2E_1 + 2E_2 + E_3 = \left[2(1)^2 + 2(2^2) + 1(3)^2\right]\frac{h^2}{8m\ell^2} = \boxed{19\frac{h^2}{8m\ell^2}}$$

12.
39-38

The shortest wavelength X-ray has the most energy, which is the maximum kinetic energy of the electron in the tube:

$$E=\frac{hc}{\lambda}=\frac{(6.63\times10^{-34}\ \mathrm{J{\cdot}s})(3.00\times10^{8}\ \mathrm{m/s})}{(1.60\times10^{-19}\ \mathrm{J/eV})(0.027\times10^{-9}\mathrm{m})}=4.6\times10^{4}\ \mathrm{eV}=46\mathrm{keV}$$

Thus the operating voltage of the tube is $\boxed{46\mathrm{kV}}$.

13.
39-42

We follow the procedure of Example 39-6, of using the Bohr formula, Eq. 37-15, with Z replaced by $Z-1$.

$$\frac{1}{\lambda}=\left(\frac{e^4m}{8\varepsilon_0^2h^3c}\right)(Z-1)^2\left(\frac{1}{n'^2}-\frac{1}{n^2}\right)=(1.097\times10^{7}\mathrm{m}^{-1})(27-1)^2\left(\frac{1}{1^2}-\frac{1}{2^2}\right)=5.562\times10^{9}\mathrm{m}^{-1}\ \rightarrow$$

$$\lambda=\frac{1}{5.562\times10^{9}\mathrm{m}^{-1}}=\boxed{1.798\times10^{-10}\mathrm{m}}$$

14.
39-47

We use Eq. 39-14 for the magnetic moment, since the question concerns spin angular momentum. The energy difference is the difference in the potential energies of the two spin states.

$$\Delta U=(\mu_z B)_{\text{spin up}}^{\text{spin down}}=-g\mu_B B\Delta m_s=-(2.0023)\frac{(9.27\times10^{-24}\ \mathrm{J/T})}{(1.60\times10^{-19}\ \mathrm{J/eV})}(2.5\mathrm{T})\left(-\tfrac{1}{2}-\tfrac{1}{2}\right)=\boxed{2.9\times10^{-4}\mathrm{eV}}$$

15.
39-48

(*a*) The energy difference is the difference in the potential energies of the two spin states. Use Eq. 39-14 for the magnetic moment.

$$\Delta U = (\mu_z B)_{\text{spin up}}^{\text{spin down}} = -g\mu_B B\Delta m_s = -(2.0023)\frac{(9.27\times10^{-24}\,\text{J/T})}{(1.60\times10^{-19}\,\text{J/eV})}(1.0\,\text{T})(-\tfrac{1}{2}-\tfrac{1}{2})$$

$$= 1.160\times10^{-4}\,\text{eV} \approx \boxed{1.2\times10^{-4}\,\text{eV}}$$

(*b*) Calculate the wavelength associated with this energy change.

$$\Delta U = E = h\frac{c}{\lambda} \rightarrow$$

$$\lambda = \frac{hc}{\Delta U} = \frac{(6.63\times10^{-34}\,\text{J}\cdot\text{s})(3.00\times10^{8}\,\text{m/s})}{(1.160\times10^{-4}\,\text{eV})(1.60\times10^{-19}\,\text{J/eV})} = 1.072\times10^{-2}\,\text{m} \approx \boxed{1.1\,\text{cm}}$$

(c) The answer would be no different for hydrogen. The splitting for both atoms is due to an *s*-state electron: 1*s* for hydrogen, 5*s* for silver. See the discussion on page 1058 concerning the Stern-Gerlach experiment.

16.
39-64

The value of ℓ can range from 0 to $n-1$. Thus for $n=6$, we have $0 \le \ell \le 5$. The magnitude of $\vec{\mathbf{L}}$ is given by Eq. 39-3, $L = \sqrt{\ell(\ell+1)}\hbar$.

$$\boxed{L_{\min} = 0\ ;\ L_{\max} = \sqrt{30}\hbar}$$