Physics 237, Spring 2008
Homework #11
Due in class, Thursday April 24, 2008

Homework is from Eisberg and Resnick (E&R) unless otherwise indicated. Please note that "questions" can be answered briefly; "problems" may be more involved. Each question usually counts 5 points; each problem counts 10 points.

1. E&R, chapter 8, problem 14.
2. E&R, chapter 8, problem 16.
3. E&R, chapter 8, problem 18.
4. E&R, chapter 9, problem 4.
5. E&R, chapter 9, problem 5.

HW#11 Solutions (Chapter 8)

(14) Verify that the parities of the one-electron atom eigenfunctions ψ_{300}, ψ_{310}, ψ_{320}, and ψ_{322} are determine by $(-1)^{\ell}$.

Under a parity operation all rectangular coordinates change sign. Under spherical coordinates we have

$$r \to r \qquad \theta \to \pi - \theta \qquad \varphi \to \pi + \varphi$$

So we must show that $\psi_{n\ell m_\ell}(r, \pi-\theta, \pi+\varphi) = (-1)^{\ell}\psi_{n\ell m}(r,\theta$
from table 7-2 (pg 243)

$$\psi_{300} = \frac{1}{81\sqrt{3\pi}}\left(\frac{Z}{a_0}\right)^{3/2}\left(27 - 18\frac{Zr}{a_0} + 2\frac{Z^2r^2}{a_0^2}\right)e^{-Zr/3a_0}$$

$$P\psi_{300} = \psi_{300} = \psi_{300}(-1)^{\ell=0} \checkmark$$

$$\psi_{310} = \frac{\sqrt{2}}{81\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2}\left(6 - \frac{Zr}{a_0}\right)\frac{Zr}{a_0}e^{-Zr/3a_0}\cos\theta$$

$$P\psi_{310} = \psi_{310}\frac{\cos(\pi-\theta)}{\cos\theta} = \psi_{310}\frac{-\cos(-\theta)}{\cos(\theta)} = -\psi_{310} = \psi_{310}(-1)$$

$$\psi_{320} = \frac{1}{81\sqrt{6\pi}}\left(\frac{Z}{a_0}\right)^{3/2}\frac{Z^2r^2}{a_0^2}e^{-Zr/3a_0}(3\cos^2\theta - 1)$$

$$P\psi_{320} = \psi_{320}\frac{(3\cos^2(\pi-\theta) - 1)}{(3\cos^2(\theta) - 1)} = \psi_{320}\frac{3(-\cos\theta)^2 - 1}{3\cos^2\theta - 1}$$

$$= \psi_{320} = \psi_{320}(-1)^{\ell=2} \checkmark$$

$$\psi_{322} = \frac{1}{162\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2}\frac{Z^2 r^2}{a_0^2} e^{-Zr/3a_0} \sin^2\theta \, e^{\pm 2i\varphi}$$

$$P\psi_{322} = \psi_{322}\frac{\sin^2(\pi-\theta)}{\sin^2\theta}\frac{e^{\pm 2i(\pi+\varphi)}}{e^{\pm 2i\varphi}}$$

$$= \psi_{322}\frac{(-\sin(-\theta))^2}{\sin^2\theta}\frac{e^{\pm 2i\varphi}}{e^{\pm 2i\varphi}} e^{\pm 2i\pi} = \psi_{322}\cos(\pm 2\pi)$$

$$= \psi_{322} = \psi_{322}(-1)^{\ell=2} \checkmark$$

(16) Show that the selection rule $\Delta\ell = \pm 1$ is valid for transitions $n=2 \to n=1$ of the hydrogen atom.

The electric dipole matrix element has the form

$$P_{fi} = \left|\int \psi_f^* e\vec{r}\,\psi_i \, d\tau\right|$$

$n=2, \ell=1, m_\ell=\pm 1 \to n=2, \ell=0, m_\ell=0$

ℓ, m_ℓ only deal with angular part of wave functic

$$P_{fi} = \left|\int R_f^*(r)\,\Theta_f^*(\theta)\,\Phi_f^*(\varphi)\,e\vec{r}\,R_i(r)\,\Theta_i(\theta)\,\Phi_i(\varphi)\,d\tau\right|$$

$$e\int_0^\infty R_f^*(r)\,R_i(r)\,r^3 dr = \text{Constant} = C$$

So we evaluate the rest of the integrals

$$I = \int_0^\pi \Theta_f^*(\theta)\,\Theta_i(\theta)\sin\theta\,d\theta \int_0^{2\pi} \frac{\vec{r}}{r}\,d\varphi\,\Phi_f^*(\varphi)\,\Phi_i(\varphi)$$

$$I_x = \int_0^{2\pi}\int_0^{\pi} \sin\theta \, d\theta \cdot \sin\theta\cos\varphi \, d\varphi$$

$$= \int_0^{\pi} \sin^2\theta \, d\theta \int_0^{2\pi} \cancel{\cos\varphi}^{\,0} \, d\varphi = 0 \checkmark$$

$$I_y = \int_0^{\pi} \sin^2\theta \, d\theta \int_0^{2\pi} \cancel{\sin\varphi\cos\varphi}^{\,0} = 0 \checkmark$$

$\Delta l \neq 0$

$$I_z = \int_0^{\pi} \cancel{\sin\theta\cos\theta}^{\,0} \int_0^{2\pi} d\varphi = 0 \checkmark$$

for $n=2, l=1, m_l=0 \rightarrow n=1, l=0, m_l=0$

$$I_x = \int_0^{2\pi}\int_0^{\pi} \cos\theta \, d\theta \, d\varphi \sin\theta\cos\varphi \sin\theta$$

$$= \int_0^{\pi} \cos\theta\sin^2\theta \, d\theta \int_0^{2\pi} \cancel{\cos\varphi}^{\,0} \, d\varphi$$

$$= 0$$

$$I_y = \int_0^{\pi} \cos\theta\sin^2\theta \, d\theta \int_0^{2\pi} \cancel{\sin\varphi\cos\varphi}^{\,0} \, d\varphi = 0$$

$$I_z = \int_0^{\pi} \cos^2\theta \sin\theta \int_0^{2\pi} d\varphi = -\cos^3\theta \Big|_0^{\pi} \cdot 2\pi$$

$= 4\pi \neq 0$ transition allowed

for $n=2, l=1, m_l=\pm1 \rightarrow n=1, l=0, m_l=0$

$$I_x = \int_0^{2\pi}\int_0^{\pi} \sin^2\theta \, e^{\pm i\varphi} \sin\theta \cos\varphi \, d\theta \, d\varphi$$

$$= \int_0^{\pi} \sin^3\theta \, d\theta \int_0^{2\pi} e^{\pm i\varphi} \cos\varphi$$

$$= \left[-\frac{1}{3}\cos\theta(\sin^2\theta+2)\right]\Big|_0^{\pi} \int_0^{2\pi} e^{\pm i\varphi}\left(\frac{e^{i\varphi}+e^{-i\varphi}}{2}\right) d\varphi$$

$$= -\frac{1}{3}(-1-(+1))(2) \int_0^{2\pi} \frac{e^{i\varphi(1\pm1)} + e^{-i\varphi(\pm1-1)}}{2} d\varphi$$

$$\overset{+1}{=} -\frac{4}{3}\left[\int_0^{2\pi} \frac{e^{2i\varphi}+1}{2} d\varphi\right]$$

$$= -\frac{2}{3}(2\pi - 0) = -\frac{4\pi}{3}$$

$$-1 = -\frac{4}{3}\left[\int_0^{2\pi} \frac{1+e^{i\varphi 2}}{2} d\varphi\right] = -\frac{4\pi}{3} \neq 0$$ transition allowed

$$I_y = \int_0^{\pi} \sin^3\theta \, d\theta \int_0^{2\pi} e^{\pm i\varphi} \sin\varphi \, d\varphi$$

$$= -\frac{4}{3}\int_0^{2\pi}\left[\frac{e^{\pm i\varphi}}{2i}(e^{i\varphi}-e^{-i\varphi}) \, d\varphi\right]$$

$$\overset{+1}{=} -\frac{4\pi i}{3} \neq 0$$

$$\overset{-1}{=} \frac{4\pi i}{3}$$

$$I_z = \int_0^{\pi} \underbrace{\sin^2\theta \cos\theta \, d\theta}_{0} \int_0^{2\pi} e^{\pm i\varphi} d\varphi = 0$$

(18) a) Calculate rate for $n=1 \rightarrow n=0$ transitions in a simple harmonic oscillator carrying charge e.

$$\mathcal{C} \doteq 10^3 \ J/m^2$$

For a harmonic oscillator

$$\Psi_0 = A_0 e^{-u^2/2} \qquad u = \underbrace{[(\mathcal{C}m)^{1/4}/\hbar^{1/2}]}_{\alpha} x$$

$$\Psi_1 = A_1 u e^{-u^2/2}$$

$$\nu = \frac{1}{2\pi}\sqrt{\frac{\mathcal{C}}{m}} \qquad du = \alpha\, dx$$

$$P_{10} = \int \Psi_1^* e x \Psi_0 \, dx$$

$$1 = \int \Psi_0 \Psi_0 dx = A_0^2 \int e^{-u^2} du \frac{1}{\alpha} = \frac{A_0^2}{\alpha}\sqrt{\pi}$$

$$A_0 = \frac{\alpha^{1/2}}{\pi^{1/4}}$$

$$1 = \int \Psi_1 \Psi_1 dx = A_1^2 \int_0^\infty u^2 e^{-u^2} \frac{du}{\alpha} = A_1^2 \left[-\tfrac{1}{2} u e^{-u^2}\right]_0^\infty - \int_0^\infty \left(-\tfrac{1}{2}\right) e^{-u^2} du$$

$$= A_1^2 \frac{\sqrt{\pi}}{4\alpha} \qquad A_1 = \frac{2\alpha^{1/2}}{\pi^{1/4}}$$

$$P_{10} = \int_0^\infty \left(\frac{\alpha^{1/2}}{\pi^{1/4}}\right)\left(\frac{2\alpha^{1/2}}{\pi^{1/4}}\right) e^{-u^2/2} \, ex \, dx \, e^{-u^2/2} u$$

$$= \frac{2e\alpha}{\sqrt{\pi}} \int_0^\infty e^{-u^2} u^2 \frac{du}{\alpha^2}$$

$$= \frac{2e\alpha}{\sqrt{\pi}} \frac{\sqrt{\pi}}{4\alpha^2} = \frac{2e}{\alpha}$$

$$P_{10}^2 = \frac{4e^2}{\alpha^2}$$

$$R = \frac{16\pi^3 \nu^3}{3\epsilon_0 h c^3} P_{fi}^2 \qquad (8\text{-}43)$$

$$= \frac{16\pi^{\cancel{3}\,2}\left(\frac{1}{2\pi}\sqrt{\frac{\mathcal{C}}{m}}\right)^3}{3\epsilon_0 h c^3}\left(\frac{4e^2\hbar}{(\mathcal{C}m)^{1/2}}\right)$$

$$= \frac{32\pi^2 \mathcal{C} e^2 \hbar}{3\epsilon_0 h c^3 m^2} = \frac{16\pi}{3}\,\frac{(10^3\ J/m^2)\left(1.6\times10^{-19}\ C\right)^2}{(3\times10^8\ m/s)^3\,\underbrace{(2\times10^{-27}\ Kg)^2}\,(8.85\times10^{-12}\ C^2N^{-1}m^{-2})}$$

$\cong 4.5\times10^5\ \frac{\text{transitions}}{\text{second}}$ varies with molecular mass

b) $\tau = \frac{1}{R} \cong 2.23\times10^{-6}\ s$

(4) Verify the eigenfunction of example 9-2 is antisymmetric with respect to a change of labels of two particles.

$$\Psi_A = \frac{1}{\sqrt{3!}}\Big[\psi_\alpha(1)\psi_\beta(2)\psi_\gamma(3) + \psi_\beta(1)\psi_\gamma(2)\psi_\alpha(3)$$
$$+\psi_\gamma(1)\psi_\alpha(2)\psi_\beta(3) - \psi_\gamma(1)\psi_\beta(2)\psi_\alpha(3)$$
$$-\psi_\beta(1)\psi_\alpha(2)\psi_\gamma(3) - \psi_\alpha(1)\psi_\gamma(2)\psi_\beta(3)\Big]$$

All combination orders of α, β, γ are present in positive and negative sign terms symmetrically for each indice. So we only have to show that Ψ_A is antisymmetric under exchange of two indices.

$$\Psi_A \overset{\alpha\leftrightarrow\beta}{\Longrightarrow} \Psi_A' = \frac{1}{\sqrt{3}}\Big[-\psi_\beta(1)\psi_\alpha(2)\psi_\gamma(3) - \psi_\beta(1)\psi_\gamma(2)\psi_\alpha(3)$$
$$+\psi_\gamma(1)\psi_\alpha(2)\psi_\beta(3) + \psi_\gamma(1)\psi_\beta(2)\psi_\alpha(3)$$
$$+\psi_\beta(1)\psi_\alpha(2)\psi_\gamma(3) + \psi_\alpha(1)\psi_\gamma(2)\psi_\beta(3)\Big] \checkmark$$

(5) If two particles are in the same state then ψ_A in 9-2 becomes

for ex. $(\alpha = \beta)$

$$\psi_A = \frac{1}{\sqrt{3}}\Big[\psi_\alpha(1)\psi_\alpha(2)\psi_\gamma(3) + \psi_\alpha(1)\psi_\gamma(2)\psi_\alpha(3) + \psi_\gamma(1)\psi_\alpha(2)\psi_\alpha(3) - \psi_\gamma(1)\psi_\alpha(2)\psi_\alpha(3) - \psi_\alpha(1)\psi_\alpha(2)\psi_\gamma(3) - \psi_\alpha(1)\psi_\gamma(2)\psi_\alpha(3) \Big] = 0 \checkmark$$