SOLUTIONS

H.W #2 PHY 237

(a) $z_1 = a_1 + ib_1$ & $z_2 = a_2 + ib_2$. Show $|z_1 z_2| = |z_1||z_2|$

$$|z_1| = \sqrt{a_1^2 + b_1^2} \quad ; \quad |z_2| = \sqrt{a_2^2 + b_2^2}$$

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i\,(b_1 a_2 + b_2 a_1)$$

$$|z_1 z_2| = \sqrt{(a_1 a_2 - b_1 b_2)^2 + (b_1 a_2 + b_2 a_1)^2}$$

$$= \sqrt{a_1^2 a_2^2 + b_1^2 b_2^2 + b_1^2 a_2^2 + b_2^2 a_1^2 + \cancel{2b_1 a_2 b_2 a_1} - \cancel{2a_1 a_2 b_1 b_2}}$$

$$= \sqrt{a_2^2(a_1^2 + b_1^2) + b_2^2(a_1^2 + b_1^2)}$$

$$= \sqrt{(a_2^2 + b_2^2)(a_1^2 + b_1^2)}$$

$$|z_1||z_2| = \sqrt{(a_1^2 + b_1^2)}\,\sqrt{(a_2^2 + b_2^2)} = \sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}$$

or $|z_1 z_2| = |z_1||z_2|$

(b)

(i) $a+ib = 2+2\sqrt{3}\,i$; $r=\sqrt{a^2+b^2}$ $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

$r=\sqrt{a^2+b^2} = \sqrt{2^2+12} = \sqrt{16} = 4$

$\theta = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

$a+ib) = r\left[\frac{2}{r} + \frac{2}{r}\sqrt{3}\,i\right] = 4\left[\frac{1}{2} + \frac{\sqrt{3}}{2}\,i\right]$

$\cos\theta = \frac{1}{2}$ & ~~Co~~ $\sin\theta = \frac{\sqrt{3}}{2}$ where the polar form is $r\left[\cos\theta + i\sin\theta\right]$

$\therefore$ $\theta = \frac{\pi}{3}$ is consistent

$a+ib = 4\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]$

(ii) $a+ib = \cancel{2+2\sqrt{3}}\ -5+5i$

$r = \sqrt{a^2+b^2} = \sqrt{25+25} = 5\sqrt{2}$

$\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}(-1) = \frac{3\pi}{4}$ or $\frac{-\pi}{4}$

$(a+ib) = r\left[\frac{-5}{r} + i\,\frac{5}{r}\right] = 5\sqrt{2}\left[\frac{-1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}\right]$

$\cos\theta = \frac{-1}{\sqrt{2}}$ & $\sin\theta = \frac{1}{\sqrt{2}}$ this is consistent with $\theta = \frac{3\pi}{4}$

but not for $\theta = \frac{-\pi}{4}$

$\therefore$ $a+ib = 5\sqrt{2}\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right]$

(iii) $a+ib = -3i$

$r = \sqrt{a^2+b^2} = \sqrt{0+(-3)^2} = 3$

$\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(-\frac{3}{0}\right) = \tan^{-1}(-\infty) = \tan^{-1}(-\infty) = -\frac{\pi}{2}, \frac{3\pi}{2}$

$a+ib = r\left[-\frac{3}{r}i\right] = 3\left[-i\right]$

$\cos\theta = 0$; $\sin\theta = -1$ this is consistent for both $\frac{3\pi}{2}$ & $-\frac{\pi}{2}$

~~but $\frac{3\pi}{2} = -\frac{\pi}{2}$~~

$a+ib = 3\left[\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right]$

(iv) $a+ib = 3.14159 = \pi$

$r = \sqrt{a^2+b^2} = \sqrt{\pi^2+0^2} = \pi$

$\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}(0) = 0, \pi, 2\pi$

$a+ib = r\left[\frac{\pi}{r}\right] = 1\left[1\right]$; $\cos\theta = 1$ $\sin\theta = 0$

This is consistent for $\theta = 0, 2\pi$ not for $\theta = \pi$

$\therefore$ $a+ib = \pi\left(\cos 2\pi + i\sin 2\pi\right)$

(c) $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ & $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

$$z_1 z_2 = r_1 r_2\left(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i\left(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1\right)\right)$$

$$= r_1 r_2\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$$

(d) (i) $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ (ii) $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{i^n \theta^n}{n!} = \sum_{n=0}^{\infty} \frac{i^{2n}\theta^{2n}}{2n!} + \sum_{n=0}^{\infty} \frac{i^{2n+1}}{2n+1!}\theta^{2n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{2n!} + i\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1!}\theta^{2n+1}$$

$$= \cos\theta + i\sin\theta$$

$$\therefore \cos\theta = \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{2n!} \quad ; \quad \sin\theta = \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n+1}}{2n+1!}$$

$$e^{-i\theta} = \sum_{n=0}^{\infty} \frac{(-i)^n\theta^n}{n!} = \sum_{n=0}^{\infty} \frac{(-i)^{2n}\theta^{2n}}{2n!} + \sum_{n=0}^{\infty} \frac{(-i)^{2n+1}}{(2n+1)!}\theta^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\theta^{2n}}{2n!} - i\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1))!}\theta^{2n+1}$$

$$= \cos\theta - i\sin\theta$$

$e^{i\theta} = \cos\theta + i\sin\theta$ & $e^{-i\theta} = \cos\theta - i\sin\theta$

$\therefore\ 2\cos\theta = e^{i\theta} + e^{-i\theta}$

or $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

$2i\sin\theta = e^{i\theta} - e^{-i\theta}$

or $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

Q.9) The mass m of a particle appears explicitly in the Schrödinger's equation, but its charge e does not, even through both may effect its motion. Why?

The Schrödinger Equation $\Rightarrow -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = \frac{\hbar}{i}\frac{\partial\psi}{\partial t}$

The first term in the equation here which corresponds to the Kinetic Energy of a particle involves mass explicity & since the particle's motion and its speed depends on how massive it is, it makes sense to have the mass m of a particle explicitly in the Schrödinger eqn. If the particle is now in a system, which has no potential it moves as a free particle, thus it definitely has some Kinetic Energy and that is where the mass of the particle comes into picture, but the charge which the particle may possess does not affect it's motion because it is moving in a potential free zone. Thus, the mass of a particle is much more fundamentally involved in determining the state of motion and is involved explicitly in the Equation, although effects due to the charge may be included in the potential V wherever relevant in determining the motion.

16.) Why does the probability density function have to be everywhere real, non negative, and of finite and definite value?

Probability can take on values between 0 and 1.

Real: Only a real function integrated over real space can yield a real value, which is the case here, since probability values vary from 0 – 1

Non negative: If we consider negative probability densities, one may end up with negative probabilities in certain regions where the density is negative and according to our first assumption probability can not be less than 0. Thus, one cannot have negative probability density or probability density function is non negative.

Finite: This criteria has to be satisfied or the integral over real space may blow up contradicting our first assumption.

17.) Explain in words what is meant by the normalization of a wave function.

From Max Born's interpretation, the probability density associated with the particle is given by $|\psi|^2$. Now, probability density integrated over all space must yield 1, since there is definite probability of finding the particle somewhere in space. Therefore;

$\int |\psi|^2 \, d\tau = 1$. This condition is the normalization condition of the wavefunction which makes the interpretation of ψ as a probability amplitude & $|\psi|^2$ as a probability density more sensible.

1.) If the wavefunctions $\Psi_1(x,t)$, $\Psi_2(x,t)$ and $\Psi_3(x,t)$ are 3 solns to the Schrödinger's equation for a particular potential $V(x,t)$, show that the arbitary linear combination $\Psi(x,t) = c_1 \Psi_1(x,t) + c_2 \Psi_2(x,t) + c_3 \Psi_3(x,t)$ is also a solution to the equation.

$$i\hbar \frac{\partial \Psi_1}{\partial t} = -\frac{\hbar^2}{2m} \frac{d^2 \Psi_1}{dx^2} + V(x,t)\,\Psi_1 \quad \text{--- (a)}$$

$$i\hbar \frac{\partial \Psi_2}{\partial t} = -\frac{\hbar^2}{2m} \frac{d^2 \Psi_2}{dx^2} + V(x,t)\,\Psi_2 \quad \text{--- (b)}$$

$$i\hbar \frac{\partial \Psi_3}{\partial t} = -\frac{\hbar^2}{2m} \frac{d^2 \Psi_3}{dx^2} + V(x,t)\,\Psi_3 \quad \text{--- (c)}$$

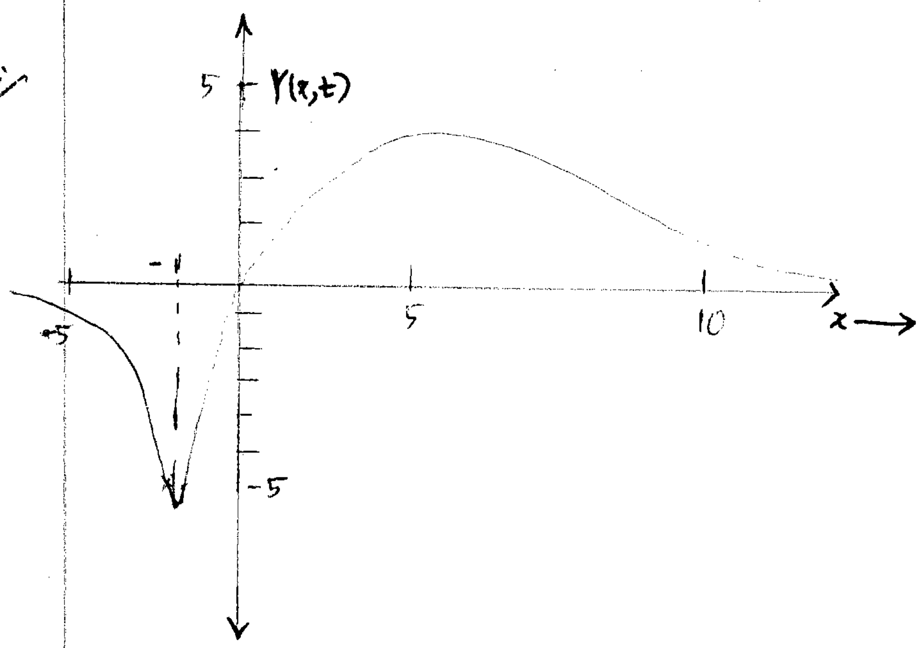
$c_1 \times (a) + c_2 \times (b) + c_3 \times (c)$

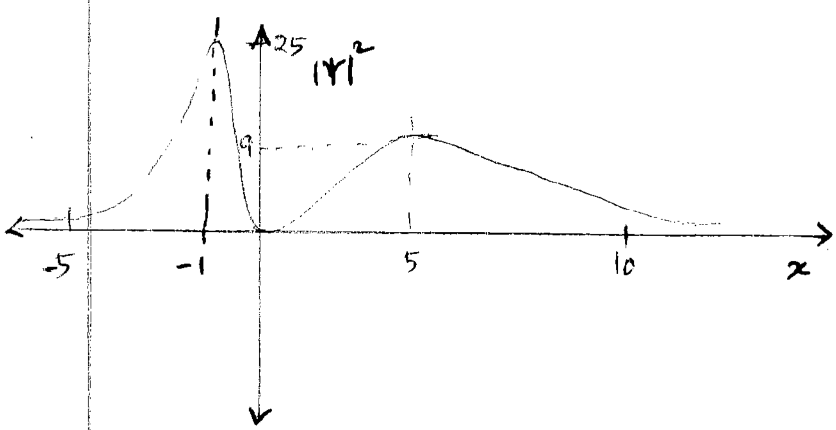
$\Rightarrow$

$$i\hbar \left[c_1 \frac{\partial \Psi_1}{\partial t} + c_2 \frac{\partial \Psi_2}{\partial t} + c_3 \frac{\partial \Psi_3}{\partial t} \right] = -\frac{\hbar^2}{2m} \left[c_1 \frac{d^2 \Psi_1}{dx^2} + c_2 \frac{d^2 \Psi_2}{dx^2} + c_3 \frac{d^2 \Psi_3}{dx^2} \right] + V(x,t) \left[c_1 \Psi_1 + c_2 \Psi_2 + c_3 \Psi_3 \right]$$

$$\Rightarrow i\hbar \frac{\partial}{\partial t} \left[c_1 \Psi_1 + c_2 \Psi_2 + c_3 \Psi_3 \right] = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \left(c_1 \Psi_1 + c_2 \Psi_2 + c_3 \Psi_3 \right) + V(x,t) \left(c_1 \Psi_1 + c_2 \Psi_2 + c_3 \Psi_3 \right)$$

2.





(b) $|\Psi|^2 \to 0$ for $x < -5$ & $x > 10$ also $|\Psi|^2 = 0$ for $x = 0$
Thus, ~~they~~ the associated particle is least likely to be found in these places

(a) $|\Psi|^2$ is maximum at $x = -1$; therefore it is most likely to be found at $x = -1$

(c) The area under the graph ($|\Psi|^2$ vs x) is greater in the positive region, thus chances are better for positive values of x.

9.)

(a)

$$\Psi(x,t) = A \sin \frac{2\pi x}{a} e^{-iEt/\hbar} \qquad -\frac{a}{2} < x < \frac{a}{2}$$

$$= 0 \qquad x < -\frac{a}{2} \text{ or } x > \frac{a}{2}$$

At $x = \pm \frac{a}{2}$, the wavefunction disappears thus the particle is strictly confined in the region $|x| < \frac{a}{2}$
Moreover, it vanishes beyond this region

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} + V\Psi = -\frac{\hbar}{i}\frac{\partial \Psi}{\partial t} = E\Psi.$$

$V=0$ for $|x| < a/2$

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} = -\frac{\hbar}{i}\frac{d\Psi}{dt}$$

R.H.S

$$-\frac{\hbar}{i}\frac{d\Psi}{dt} = -\frac{\hbar}{i} A \sin\frac{2\pi x}{a} \times \left(-\frac{iE}{\hbar}\right) \times e^{-iEt/\hbar}$$

$$\frac{\hbar}{i}\frac{d\Psi}{dt} = -\left(A \sin\frac{2\pi x}{a}\right)E \times e^{-\frac{iEt}{\hbar}} = +E\left(A \sin\frac{2\pi x}{a}\right) e^{-iEt/\hbar}$$

L.H.S

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} = -\frac{\hbar^2}{2m}\left[+\left(\frac{2\pi}{a}\right)\right]^2 A \sin\frac{2\pi x}{a} e^{-iEt/\hbar} = \frac{2\pi^2\hbar^2}{ma^2} A \sin\frac{2\pi x}{a} e^{-\frac{iEt}{\hbar}}$$

$$E = \frac{\langle p^2 \rangle}{2m}$$

For $\psi = A \sin \frac{2\pi x}{a} \, e^{-iEt/\hbar}$

$$\langle p^2 \rangle = \frac{\int \psi^* \left(-i\hbar \frac{\partial}{\partial x}\right)^2 \psi \, dx}{\int |\psi|^2 \, dx}$$

$$= \frac{\int A^2 \sin^2 \frac{2\pi x}{a} \times \left(\frac{2\pi}{a}\right)^2 \hbar^2 dx}{\int A^2 \sin^2 \frac{2\pi x}{a} \, dx}$$

$$= \left(\frac{2\pi}{a}\right)^2 \hbar^2 \frac{\int A^2 \sin^2 \frac{2\pi x}{a} \, dx}{\int A^2 \sin^2 \frac{2\pi x}{a} \, dx}$$

$$= \left(\frac{2\pi}{a}\right)^2 = \langle p^2 \rangle$$

$$\therefore \quad E = \left(\frac{2\pi}{a}\right)^2 \times \frac{1}{2m} = \frac{2\pi^2 \hbar}{ma^2}$$

$$\therefore \quad \text{L.H.S} \Rightarrow \frac{2\pi^2\hbar^2}{ma^2} \left(A \sin \frac{2\pi x}{a} \, e^{-iEt/\hbar}\right)$$

$$= E\left(A \sin \frac{2\pi x}{a} \, e^{-iEt/\hbar}\right)$$

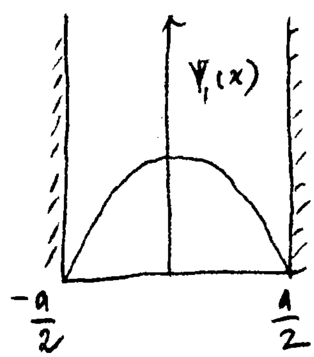
$\therefore$ L.H.S = R.H.S

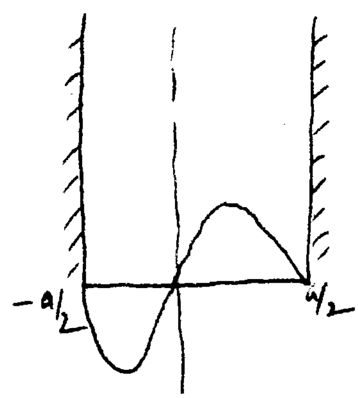
(b) As determined above or before;

$$E = \frac{2\pi^2 \hbar^2}{a^2}$$

Note this energy is 4 times greater than the ground state energy E_0

(c)





Both of these wavefunctions vanish at the boundaries $x = \pm \frac{a}{2}$. $\psi_1(x)$ is even under reflection

$$\psi_1(x) = \psi_1(-x)$$

$\psi_2(x)$ is odd under reflection. $\psi_2(-x) = -\psi_2(x)$.
Since $\psi_2(x)$ has more oscillation than $\psi_1(x)$; the energy eigenvalue corresponding to ψ_2 is larger than that of ψ_1.

$E_2 > E_1$

10)

(a)

$$\int |\Psi|^2 dx = \int A^2 \sin^2 \frac{2\pi x}{a}\, dx$$

$$= A^2 \int_{-a/2}^{a/2} \frac{\left(1 - \cos \frac{4\pi x}{a}\right)}{2}\, dx$$

$$= \frac{A^2}{2}\left\{\int_{-a/2}^{a/2} dx - \int_{-a/2}^{a/2} \cos \frac{4\pi x}{a}\, dx\right\}$$

$$= \frac{A^2}{2}\left\{ a - \left. \frac{\sin \frac{4\pi x}{a}}{\frac{4\pi}{a}} \right|_{-a/2}^{a/2} \right\}$$

$$= \frac{A^2 a}{2}$$

$$\int |\Psi|^2 dx = 1 \quad \therefore \quad \frac{A^2 a}{2} = 1$$

$$\therefore A^2 = \left(\sqrt{\frac{2}{a}}\right)^2 \text{ or } A = \sqrt{\frac{2}{a}}$$

(b) This equals the value of A for ground state wavefunctions and, in fact, the normalization constant of all the excited states equals this also. Since all space wavefunctions are simple sines & cosines, this equality is understandable.