Physics 237, Spring 2008
Homework #8
Due in class, Thursday April 3, 2008

Homework is from Eisberg and Resnick (E&R) unless otherwise indicated. Please note that "questions" can be answered briefly; "problems" may be more involved. Each question usually counts 5 points; each problem counts 10 points.

1. E&R, chapter 7, **question** 9.
2. E&R, chapter 7, problem 5.
3. E&R, chapter 7, problem 7.
4. E&R, chapter 7, problem 8.
5. E&R, chapter 7, problem 10.

·) The phenomenon of having the same quantum no: (say n) even though the other quantum nos corresponding to n are different. The system is said to be degenerate in the quantum no: n.

$Ex \rightarrow$:

$$n \rightarrow \quad E_n = -\frac{13.6}{n^2} \qquad \begin{array}{l} l = 0, 1, \dots n-1 \\ m = -l \text{ to } l. \end{array}$$

The Energy eigenvalues are degenerate.

5.
$$\psi_{100} = \frac{1}{\sqrt{\pi}} \left(\frac{Z}{a_0} \right)^{3/2} e^{-Zr/a_0}.$$

$l \; m_l = 0$

The eqn (7.17) of the text becomes,

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \frac{2\mu}{\hbar^2} \left[E - V(r) \right] R = 0$$

$R = \psi_{100}$

$$\psi_{100}(r,\theta,\phi) = \frac{1}{\sqrt{\pi a_0^3}}\, e^{-r/a_0}\,.$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi_{100} + V(r)\,\psi_{100} = E_1\,\psi_{100}$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi_{100} = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)\frac{e^{-r/a_0}}{\sqrt{\pi a_0^3}}$$

$$= \frac{1}{r^2}\frac{d}{dr}\left(r^2\left(-\frac{1}{a_0}\right)e^{-r/a_0}\right)\cdot\frac{1}{\sqrt{\pi a_0^3}}$$

$$= \frac{1}{r^2}\left[2r\left(-\frac{1}{a_0}\right) + r^2\left(-\frac{1}{a_0}\right)^2\right)e^{-r/a_0}\,\frac{1}{\sqrt{\pi a_0^3}}$$

$$= \left(\frac{1}{a_0^2} - \frac{2}{r a_0}\right)\psi_{100}\,; \quad \text{using } a_0 = \frac{4\pi\epsilon_0\hbar^2}{\mu e^2}$$

$$-\frac{\hbar^2}{2m}\left[\frac{1}{a_0^2} - \frac{2}{r a_0}\right) - \frac{e^2}{4\pi\epsilon_0 r}\right]\psi_{100} = -\frac{\hbar^2}{2ma^2}\psi_{100}$$

$$-\frac{\hbar^2}{2m}\frac{m^2e^4}{(4\pi\epsilon_0)^2\hbar^4} = \frac{me^4}{2(4\pi\epsilon_0)^2\hbar^2} = E_1.$$

) From the diagram on pg 245 of the text (R&E), one can see that the probability is maximum for $r = \dfrac{4a_0}{Z}$ where a_0 is the Bohr radius and Z is the atomic number.

$$\bar{r} = \int_0^\infty \int_0^\pi \int_0^{2\pi} r^2 dr \sin\theta d\theta d\phi \quad \psi^*_{211} \psi_{211} \, r$$

$$= \int_0^\infty \int_0^\pi \int_0^{2\pi} r^2 dr \sin\theta d\theta \, d\phi \left(\frac{1}{64\pi} \left(\frac{Z}{a_0}\right)^3 \left(\frac{Z}{a_0}\right)^2 r^2 \; e^{-Zr/a_0} \right.$$

$$\left. \times \; r \sin^2\theta \right)$$

$$= \int_0^\infty \int_0^\pi \int_0^{2\pi} r^5 \, dr \quad \sin^3\theta \, d\theta \quad d\phi \left(\frac{1}{64\pi} \left(\frac{Z}{a_0}\right)^5 e^{-Zr/a_0} \right)$$

$$= \frac{1}{64\pi} \left(\frac{Z}{a_0}\right)^5 \int_0^\infty e^{-Zr/a_0} r^5 dr \int_0^\pi \sin^3\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= \frac{1}{64\pi} \left(\frac{Z}{a_0}\right)^5 \int_0^\infty e^{-Zr/a_0} r^5 dr \int_0^\pi \sin^3\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= \frac{2\pi}{64\pi}\left(\frac{Z}{a_0}\right)^5 \int_0^\infty e^{-Zr/a_0}\, r^5\, dr \int_0^\pi \left(1 - \cos^2\theta\right) \sin\theta\, d\theta$$

$$= \frac{2\pi}{64\pi}\left(\frac{Z}{a_0}\right)^5 \int_0^\infty e^{-Zr/a_0}\, r^5\, dr \left(-\cos\theta \Big|_0^\pi + \frac{\cos^3\theta}{3}\Big|_0^\pi\right)$$

$$= \frac{2\pi}{64\pi}\left(\frac{Z}{a_0}\right)^5 \int_0^\infty e^{-Zr/a_0}\, r^5\, dr \left(2 - \frac{2}{3}\right)$$

$$= \frac{2\pi}{64\pi} \times \left(\frac{Z}{a_0}\right)^5 \times \frac{4}{3} \times \int_0^\infty e^{-Zr/a_0}\, r^5\, dr$$

$$\frac{Zr}{a_0} = y \quad , \quad dy = \frac{Z}{a_0}\, dr$$

$$= \frac{2\pi}{64\pi} \times \left(\frac{Z}{a_0}\right)^5 \times \frac{4}{3} \times \left(\frac{a_0}{Z}\right)^6 \times \int_0^\infty e^{-y}\, y^5\, dy$$

$$= \frac{\cancel{2\pi}}{\cancel{64\pi}_{\cancel{32}\,\cancel{8}}} \times \cancel{\left(\frac{Z}{a_0}\right)^5} \times \frac{\cancel{4}}{\cancel{3}} \times \cancel{\left(\frac{a_0}{Z}\right)^6} \times \overset{5}{\cancel{120}} = 5\left(\frac{a_0}{Z}\right)$$

$$5\left(\frac{a_0}{Z}\right)$$

c) The physical significance of the difference in answers for (a) & (b) lies in the fact that the first result which also coincides with the classical results from Bohr's Correspondence Principle is just the point at which the probability is maximum. The second result is the average position weighted by the probability. The first result was the classical result, the probability is maximum at the classical result, but because of the wave nature of the particle there is no definite possibility of the particle at the, ~~...~~ pt but only a probability. The mean value not coinciding with the maximum value of probability is because of the wave nature of the particle.

3.) The expectation value of $\bar{V}$ for the potential energy in the ground state is given by;

$$V = \frac{-Ze^2}{4\pi\epsilon_0 r} \qquad \bar{V} = \int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} \psi^*_{100}\, V\, \psi_{100}\; r^2 dr \sin\theta\, d\theta\, d\phi$$

$$\bar{V} = \int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} r^2 dr \sin\theta d\theta\, d\phi \left(\frac{1}{\pi}\left(\frac{Z}{a_0}\right)^3 e^{-2Zr/a_0}\left(\frac{-e^2 Z}{4\pi\epsilon_0 r}\right)\right)$$

$$= \frac{4\pi}{\pi}\int_0^{\infty} r^2 dr \left(\frac{Z}{a_0}\right)^3 e^{-\frac{2Zr}{a_0}} \left(\frac{-e^2 Z}{4\pi\epsilon_0}\right)\frac{1}{r}$$

$$= -4\left(\frac{Ze^2}{4\pi\epsilon_0}\right)\int_0^\infty r\left(\frac{Z}{a_0}\right)^3 e^{-2Zr/a_0}\,dr$$

$$= \frac{-Ze^2}{\pi\epsilon_0}\int_0^\infty r\left(\frac{Z}{a_0}\right)^3 e^{-2Zr/a_0}\,dr$$

$$= \frac{-Ze^2}{\pi\epsilon_0}\times\left(\frac{Z}{a_0}\right)^3\times\int_0^\infty r\, e^{-2Zr/a_0}\,dr$$

$$\frac{Z}{a_0}r = y \qquad \frac{Z}{a_0}dr = dy$$

$$= \frac{-Ze^2}{\pi\epsilon_0}\times\left(\frac{Z}{a_0}\right)^3\times\left(\frac{a_0}{Z}\right)^2\int_0^\infty y\, e^{-2y}\,dy$$

$$= \frac{-Ze^2}{\pi\epsilon_0}\times\left(\frac{Z}{a_0}\right)\times\frac{1}{4}$$

$$= -\frac{Z^2e^2}{4\pi\epsilon_0 a_0} \qquad \text{where } a_0 = \frac{4\pi\epsilon_0\hbar^2}{\mu e^2}$$

$$= \frac{-Z^2e^4\mu}{(4\pi\epsilon_0)^2\hbar^2}$$

$$= 2\frac{(-Z^2e^4\mu)}{(4\pi\epsilon_0)^2 2\hbar^2} = 2E$$

or $\frac{\overline{V}}{2} = E$

$\therefore \quad E = K + V$

or $E = \bar{K} + \bar{V}$ or $E = \bar{K} + \bar{V}$

or $E = +\frac{\bar{V}}{2}$ or $+\frac{\bar{V}}{2} = \bar{K} + \bar{V}$

$\therefore \quad -\frac{\bar{V}}{2} = \bar{K}$

b)
$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{2\mu}{\hbar^2}\left[E - V(r)\right] = \frac{l(l+1)R}{r^2}$$

We are attempting to find a solution for $r \to 0$

Multiplying throughout with r^2 one has;

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{2\mu}{\hbar^2}\left[E - V(r)\right]r^2 = l(l+1)R$$

low for $r \to 0$

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{2\mu}{\hbar^2}\left[E - V(r)\right] r^2 = l(l+1)R$$ leads to

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = l(l+1)R.$$ If $R(r) = r^l$ one has;

$$\frac{d}{dr}\left(r^2 l r^{l-1}\right) = \frac{d}{dr}\left(l r^{l+1}\right) = l(l+1) r^l.$$

$$= l(l+1)R.$$

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = l(l+1)R.$$