Physics 237, Spring 2005

Midterm Exam

Thursday, March 17, 2005

- Answer all questions, using the paper provided. *Begin each problem on a separate sheet.* The pages will be separated for grading, so clearly label the problem number and put your name on EACH PAGE.
- One 8.5 by 11 inch cheat sheet (front and back) allowed. No other books, calculators or notes allowed.
- SHOW ALL WORK.
- Potentially useful information:

$$d^3x = r^2 dr \sin\theta \, d\theta \, d\phi; \quad 0 \le \theta \le \pi; \;\; 0 \le \phi \le 2\pi$$

$$\int_0^{+\infty} e^{-ax^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} \qquad \int_0^{+\infty} x^2\, e^{-ax^2}\, dx = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}$$

$$\int_0^{+\infty} x\, e^{-ax^2}\, dx = \frac{1}{2a}$$

$$\int x\, e^{ax}\, dx = \frac{e^{ax}}{a^2}(ax-1) \qquad \int x^2\, e^{ax}\, dx = \frac{x^2\, e^{ax}}{a} - \frac{2}{a}\int x\, e^{ax}\, dx$$

$$\int (\sin^2 ax)\, dx = \frac{x}{2} - \frac{1}{4a}\sin 2ax \qquad \int (\cos^2 ax)\, dx = \frac{x}{2} + \frac{1}{4a}\sin 2ax$$

$$\int (\sin ax)\,(\cos ax)\, dx = \frac{1}{2a}\sin^2 ax \qquad \int (\sin^2 ax)\,(\cos^2 ax)\, dx = \frac{x}{8} - \frac{1}{32a}\sin 4ax$$

$$\int (\sin ax)\,(\cos^m ax)\, dx = -\frac{\cos^{m+1} ax}{(m+1)a} \qquad \int (\sin^m ax)\,(\cos ax)\, dx = \frac{\sin^{m+1} ax}{(m+1)a}$$

1. (20 points total) *Short answer.*

   (a) (2 points) What are the units of $\hbar$?

   (b) (9 points) Give one example of the evidence that led to the development of each of the following aspects of quantum mechanics:

      i. Wave-like behavior of particles.

      ii. Particle-like behavior of waves.

      iii. Quantization.

   (c) (3 points) What does it mean for a function to have definite parity?

   (c) (3 points) Draw an example of a potential in one dimension that has solutions with definite parity.

   (d) (3 points) Draw an example of a potential in one dimension that does *not* have solutions with definite parity.

2. (10 points total) Consider the one-dimensional potential $V(x) = C|x|$:

   (a) (2 points) Does this potential have bound states only, free (continuum) states only, or both?

   (b) (8 points) Sketch the wave functions and probability densities for the ground state and first excited state of this potential.

3. (25 points) Consider the simple harmonic oscillator with potential $V(x) = (C/2)x^2$. The ground state wave function is

$$\Psi(x,t) = A_0 \exp\left(-u^2/2\right) \exp\left(-iEt/\hbar\right)$$

   where $u = [(Cm)^{1/4}/\hbar^{1/2}]x$ and $E = h\nu/2$ where ν is the classical frequency of the oscillator.

   (a) Calculate the following expectation values:

      i. (6 points) $\bar{x}$

      ii. (6 points) $\bar{x^2}$

      iii. (6 points) $\bar{p^2}$

   (b) (7 points) Using these expectation values, show that the sum of the expectation values of the kinetic and potential energies is equal to the total energy.

4. (25 points total) Consider an electron of mass m in a one-dimensional box, with infinite square well potential $V(x) = 0$ for $|x| < a/2$ and $V(x) = \infty$ for $|x| \geq a/2$.

   (a) (8 points) Find the energy levels of this system.

   (b) (8 points) Find the wave functions for the ground state and first excited state of this system. You may leave an arbitrary overall normalization constant.

   (c) (9 points) Now let the potential be finite: $V(x) = 0$ for $|x| < a/2$ and $V(x) = V_0$ for $|x| \geq a/2$. For energy $E < V_0$, write down the wave function in each region and the conditions it must satisfy at the boundaries. Do *not* solve the equations.

5. (20 points total)

   (a) Recall that in developing the Schrödinger equation we used the plane wave solution for a free particle to figure out what operators would reproduce the energy equation.

   i. (5 points) Write down the momentum operator p_{op} and show that it satisfies

$$p_{op}\,\Psi = p\,\Psi$$

   where Ψ is the wave function for a free particle in one space dimension.

   ii. (5 points) The momentum operator you obtained is a function of the position variable x. It is sometimes convenient to think of everything as a function of momentum rather than position. Assuming momentum is the fundamental variable, now find a *position* operator x_{op} that satisfies

$$x_{op}\,\Psi = x\,\Psi$$

   where again Ψ is again the wave function for a free particle in one space dimension.

   (b) (10 points) Consider the time-independent Schrödinger equation in two dimensions. Show that the equation is separable and the wave function can be written $\psi = X(x)Y(y)$ if the potential has the form $V(x, y) = V_x(x) + V_y(y)$.