Physics 237, Spring 2008
Midterm Exam
Thursday, March 20, 2008

- Answer all questions, using the paper provided. *Begin each problem on a separate sheet.* The pages will be separated for grading, so clearly label the problem number and put your name on EACH PAGE.
- One 8.5 by 11 inch cheat sheet (front and back) allowed. No other books, calculators or notes allowed.
- SHOW ALL WORK.
- Potentially useful information:

$$d^3x = r^2 dr \sin\theta \, d\theta \, d\phi; \quad 0 \le \theta \le \pi; \;\; 0 \le \phi \le 2\pi$$

$$\int_0^{+\infty} e^{-ax^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} \qquad \int_0^{+\infty} x^2\, e^{-ax^2}\, dx = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}$$

$$\int_0^{+\infty} x\, e^{-ax^2}\, dx = \frac{1}{2a}$$

$$\int x\, e^{ax}\, dx = \frac{e^{ax}}{a^2}(ax-1) \qquad \int x^2\, e^{ax}\, dx = \frac{x^2\, e^{ax}}{a} - \frac{2}{a}\int x\, e^{ax}\, dx$$

$$\int (\sin^2 ax)\, dx = \frac{x}{2} - \frac{1}{4a}\sin 2ax \qquad \int (\cos^2 ax)\, dx = \frac{x}{2} + \frac{1}{4a}\sin 2ax$$

$$\int (\sin ax)(\cos ax)\, dx = \frac{1}{2a}\sin^2 ax \qquad \int (\sin^2 ax)(\cos^2 ax)\, dx = \frac{x}{8} - \frac{1}{32a}\sin 4ax$$

$$\int (\sin ax)(\cos^m ax)\, dx = -\frac{\cos^{m+1} ax}{(m+1)a} \qquad \int (\sin^m ax)(\cos ax)\, dx = \frac{\sin^{m+1} ax}{(m+1)a}$$

1. (25 points total) *Short answer.*

   (a) (2 points) What are the units of $\hbar$?

   (b) (8 points) Give an example of a physical phenomenon that led to the development of quantum mechanics. Specify at least one classical prediction for the system that was wrong, and identify the actual result (corresponding to the prediction) that quantum mechanics explained.

   (c) (9 points) Give an example of a potential that gives rise to the following.

   i. No bound states — continuum energies only.

   ii. Bound states (quantized energies) only – no continuum.

   iii. Both bound and continuum states.

   (d) (6 points)

   i. Let Ψ be the solution to the *one* (space) dimensional Schrödinger equation. What units must it have?

   ii. Now let Ψ be the solution to the *three* dimensional Schrödinger equation. What units must it have?

2. (20 points total) Consider the following potential: $V(x) = 0$ for $x < 0$ and $V(x) = \infty$ for $x \geq 0$, i.e. there is an impenetrable wall at $x = 0$. Consider a particle of mass m and energy E incident from $x < 0$.

   (a) (10 points) Find the solution to the Schrödinger equation for this particle. You may leave the overall normalization (i.e., one complex constant) arbitrary.

   (b) (10 points) Find the reflection and transmission coefficients at the wall.

3. (25 points total) Shortly after Schrödinger came up with his nonrelativistic equation, Dirac tried to do the same thing for relativistic quantum mechanics. The difference is that the energy-momentum relation to be satisfied is

$$p^2c^2 + m^2c^4 = E^2$$

   where c is the speed of light.

   (a) (15 points) Using the energy and momentum operators, find an equation for the wave function Ψ (in one space dimension) that satisfies the relativistic energy-momentum relation. Just consider a free particle; do not put in a potential.

   (b) (10 points) Separate the equation into two separate equations for the space and time dependence. (Hint: keep the m^2c^4 term with the space dependent part.)

4. (10 points total) Consider an electron of mass m in a one-dimensional box of width a, with infinite square well potential $V(x) = 0$ for $|x| < a/2$ and $V(x) = \infty$ for $|x| \geq a/2$.

   (a) (3 points) If the electron has momentum p, what it its de Broglie wavelength?

   (b) (7 points) *Without* actually solving the Schrödinger equation, find the energy levels of the electron by requiring that half-integral numbers of de Broglie wavelengths fit into the box.

5. (20 points total) Consider the simple harmonic oscillator with potential $V(x) = (C/2)x^2$. The ground state wave function is

$$\Psi(x,t) = A_0 \exp\left(-u^2/2\right) \exp\left(-iEt/\hbar\right)$$

   where $u = \sqrt{\alpha}\, x$, $\quad E = \frac{\hbar}{2}\sqrt{\frac{C}{m}}$, $\quad |A_0|^2 = \sqrt{\alpha/\pi}$, and $\quad \alpha = \sqrt{Cm}/\hbar$.

   (a) (7 points) Calculate the expectation value of x^2.

   (b) (7 points) Calculate the expectation value of p^2.

   (c) (6 points) Using these expectation values, show that the sum of the expectation values of the kinetic and potential energies is equal to the total energy.

Problem 1

Midterm Exam
Spring 2008
Solutions

a) units of $\hbar$: $[\hbar]$ = joule-sec = $\frac{kg\ m^2}{sec}$

b) Physical phenomena that classical predictions got wrong but QM got right. There are many. Some examples:

- Blackbody radiation: thermal radiation from a body at a specific temp.

Classical pred.	Exptal + QM reality
• total energy in vol. infinite	• total energy finite: $R_{TOT} = \sigma T^4$
• spectrum rises indefinitely w/ ν	• spectrum peaks at finite $\nu \propto T$

- Photoelectric effect: emission of electrons when UV light shone on metal

Class. pred	QM reality
• higher light intensity ⇒ higher KE electrons	• higher intensity ⇒ more electrons, same KE
• effect happens for any ν	• no effect below cutoff freq. ν_0
• KE of electrons indep of ν	• higher ν ⇒ higher KE

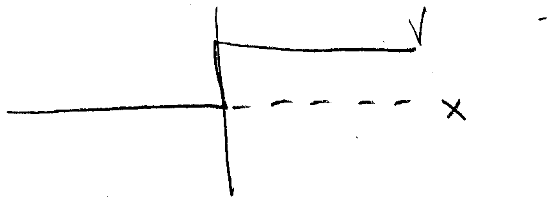
- Atomic spectra + stability: electrons in atoms

Class. pred	QM reality
• electron should spiral into nucleus	• atom stable
• emission spectrum continuous	• spectrum discrete

Problem 1, cont

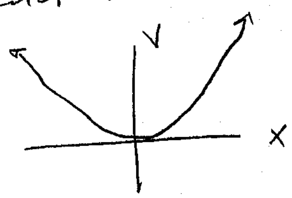
c) Potentials. Again, many possible correct answers

i) No bound states



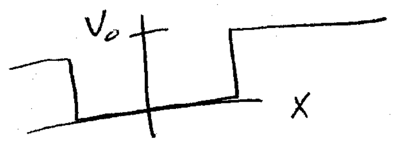
or any potential where E can always be $> V$ as $x \to +\infty$ <u>or</u> as $x \to -\infty$

ii) Bound states only



or any potential that $\to \infty$ as $x \to +\infty$ <u>and</u> as $x \to -\infty$

iii) Both bound and continuum states



$0 < E < V_0$ bound

$E > V_0$ continuum

or any potential that has a dip but V finite as $x \to +\infty$ or $x \to -\infty$

d) Units of Ψ: use normalization condition

i) 1D: $1 = \int \Psi^* \Psi \, dx \Rightarrow [1] = [\Psi]^2 m \Rightarrow [\Psi]^2 = \frac{1}{m}$

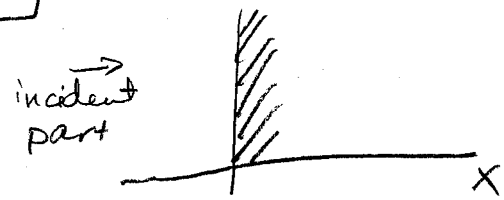
$$\Rightarrow [\Psi] = m^{-1/2} = \frac{1}{\sqrt{\text{length}}}$$

ii) 3D: $1 = \int \Psi^* \Psi \, d^3x \Rightarrow [1] = [\Psi]^2 m^3 \Rightarrow [\Psi]^2 = \frac{1}{m^3}$

$$\Rightarrow [\Psi] = m^{-3/2} = \left(\frac{1}{\text{length}}\right)^{3/2}$$

Problem 2

$$V(x) = \begin{cases} 0 & x < 0 \\ \infty & x \geq 0 \end{cases}$$



Particle mass m, energy E incident fr/ $x < 0$

a) Find wave function ψ

$\underline{x < 0}$: $\quad -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$

$$\Rightarrow \psi(x) = Ae^{+ikx} + Be^{-ikx}, \quad k = \frac{\sqrt{2mE}}{\hbar} \quad \text{fr/ subst. into S.E.}$$

$\underline{x \geq 0}$: wall is impenetrable, so

$$\psi(x) = 0, \quad x \geq 0$$

Boundary condition at $x = 0$ is

ψ continuous at $x = 0 \Rightarrow$

$$\psi(0) = A + B = 0 \Rightarrow B = -A$$

(and $\frac{d\psi}{dx}$ is not continuous because $V \to \infty$)

$$\therefore \quad \boxed{\psi(x) = \begin{cases} A\left(e^{+ikx} - e^{-ikx}\right) & x < 0, \quad k = \frac{\sqrt{2mE}}{\hbar} \\ 0 & x \geq 0 \end{cases}}$$

b) Find reflection & transmission coefficients at the wall,

incident wave : Ae^{+ikx}

reflected wave : $-Ae^{-ikx}$

transmitted wave : 0

Since the wall is impenetrable, we expect $R=1$ & $T=0$

$$R = \frac{|\vec{J}_{refl}|}{|\vec{J}_{inc}|} = \frac{\frac{k}{m} A^*A}{\frac{k}{m}(-A^*)(-A)} = 1$$

$$T = \frac{|\vec{J}_{trans}|}{|\vec{J}_{inc}|} = 0 \quad \text{since } \psi(x>0)=0$$

Problem 3

Relativistic energy-momentum relation

$$p^2c^2 + m^2c^4 = E^2$$

a) Use energy + momentum operators to find eq'n for Ψ in 1D.

$$P_{op} = -i\hbar\frac{\partial}{\partial x} \Rightarrow P_{op}^2 = -\hbar^2\frac{\partial^2}{\partial x^2}$$

$$E_{op} = i\hbar\frac{\partial}{\partial t} \Rightarrow E_{op}^2 = -\hbar^2\frac{\partial^2}{\partial t^2}$$

To get eq'n for Ψ, take energy rel'n to be an operator eq'n acting on Ψ:

$$E_{op}^2\,\Psi = P_{op}^2\,c^2\,\Psi + m^2c^4\,\Psi$$

or $\boxed{-\hbar^2\frac{\partial^2\Psi}{\partial t^2} = -\hbar^2c^2\frac{\partial^2\Psi}{\partial x^2} + m^2c^4\,\Psi}$, $\Psi = \Psi(x,t)$

b) Separate space + time dependence. Write $\Psi(x,t) = \psi(x)\phi(t)$

$$\Rightarrow \left(-\hbar^2\frac{d^2\phi}{dt^2}\right)\psi = -\left(\hbar^2c^2\frac{d^2\psi}{dx^2}\right)\phi + m^2c^2\psi\phi$$

+ dividing through by $\Psi = \psi\phi$

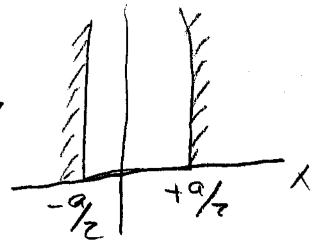
$$\underbrace{-\hbar^2\frac{d^2\phi}{dt^2}\frac{1}{\phi}}_{\text{fn of } t \text{ only}} = \underbrace{-\hbar^2c^2\frac{d^2\psi}{dx^2}\frac{1}{\psi} + m^2c^4}_{\text{fn of } x \text{ only}}$$

LHS = fn of t only; RHS = fn of x only. For LHS = RHS for all x, t, both sides must be separately equal to a const. Call it "E^2"

$\boxed{-\hbar^2\frac{d^2\phi}{dt^2} = E^2\phi}$ $\boxed{-\hbar^2c^2\frac{d^2\psi}{dx^2} + m^2c^4\psi = E^2\psi}$

Problem 4

Electron, mass m in 1-D box



a) Eletron has momentum p. What is its de Broglie wavelength?

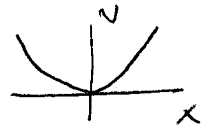
$$\boxed{\lambda = h/p}$$

b) Find energies by requiring half-integer multiples of λ fit into box.

$$\Rightarrow \left(\frac{n}{2}\right)\lambda = a$$

$$\Rightarrow \frac{n}{2}\frac{h}{p} = a \quad \Rightarrow \quad p = \frac{nh}{2a} = \frac{n\pi\hbar}{a}$$

$$E = \frac{p^2}{2m} = \frac{\pi^2\hbar^2}{2ma^2} n^2$$ = energy of particle in box

Simple harmonic oscillator $V = \frac{1}{2} c x^2$ 

(5.1)

Ground state

$$\Psi(x,t) = A_0 e^{-u^2/2} e^{-iEt/\hbar}.$$

$$u = \sqrt{\alpha}\, x \qquad E = \frac{\hbar}{2}\sqrt{\frac{c}{m}}$$

$$|A_0|^2 = \sqrt{\frac{\alpha}{\pi}}$$

$$\alpha = \frac{\sqrt{cm}}{\hbar}$$

a) Find $\overline{x^2}$

$$\overline{x^2} = \int_{-\infty}^{\infty} \Psi^* x^2 \Psi \, dx = |A_0|^2 \underbrace{\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx}_{2 * \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}}$$

fr/ integral on front

$$= \sqrt{\frac{\alpha}{\pi}} \frac{1}{2} \sqrt{\frac{\pi}{\alpha^3}} = \frac{1}{2\alpha} = \boxed{\frac{\hbar}{2\sqrt{cm}} = \overline{x^2}}$$

Check units:

$$[\overline{x^2}] = \frac{kg\, m^2}{sec} \frac{1}{[c]^{1/2}\, kg^{1/2}}$$

units for c: $\frac{kg\, m^2}{sec^2} = [V] = [c]\, m^2$

$$\Rightarrow [c] = \frac{kg}{sec^2}$$

$$\text{so } [\overline{x^2}] = \frac{kg\, m^2}{sec} \frac{1}{\left(\frac{kg\, kg}{sec^2}\right)^{1/2}}$$

$$= \frac{kg\, m^2}{sec} \frac{sec}{kg} = m^2 \text{ ok}$$

b) Find $\overline{p^2}$

$$\overline{p^2} = \int_{-\infty}^{\infty} \Psi^* \left[\left(-i\hbar \frac{\partial}{\partial x}\right)^2 \Psi \right] dx$$

$$\frac{\partial \Psi}{\partial x} = \frac{\partial}{\partial x} A_0 e^{-\alpha x^2/2} e^{-iEt/\hbar} = -\alpha x \Psi$$

$$\frac{\partial^2 \Psi}{\partial x^2} = -\alpha \Psi + \alpha^2 x^2 \Psi$$

$$\Rightarrow \overline{p^2} = -\hbar^2 \left[-\alpha \underbrace{\int_{-\infty}^{\infty} \Psi^* \Psi \, dx}_{=1} + \alpha^2 \underbrace{\int_{-\infty}^{\infty} x^2 \Psi^* \Psi \, dx}_{\overline{x^2} = \frac{1}{2\alpha} \text{ from a)}} \right]$$

$$= \hbar^2 \alpha - \frac{\hbar^2 \alpha^2}{2\alpha} = \frac{\hbar^2 \alpha}{2} = \frac{\hbar^2 \sqrt{cm}}{2\hbar} = \boxed{\frac{\hbar\sqrt{cm}}{2} = \overline{p^2}}$$

$$\text{units}: [\overline{p^2}] = \frac{\text{kg m}^2}{\text{sec}} \left(\frac{\text{kg kg}}{\text{sec}^2}\right)^{1/2} = \left(\frac{\text{kg m}}{\text{sec}}\right)^2 \text{ ok}$$

c) Show $\overline{KE} + \overline{PE} = E$

$$\overline{KE} + \overline{PE} = \frac{\overline{p^2}}{2m} + \frac{1}{2} c \overline{x^2} = \frac{1}{2m} \frac{\hbar\sqrt{cm}}{2} + \frac{1}{2} c \frac{\hbar}{2\sqrt{cm}}$$

$$= \frac{1}{4} \hbar \sqrt{\frac{c}{m}} + \frac{1}{4} \hbar \sqrt{\frac{c}{m}}$$

$$= \frac{\hbar}{2} \sqrt{\frac{c}{m}} = E \quad \text{ok}$$