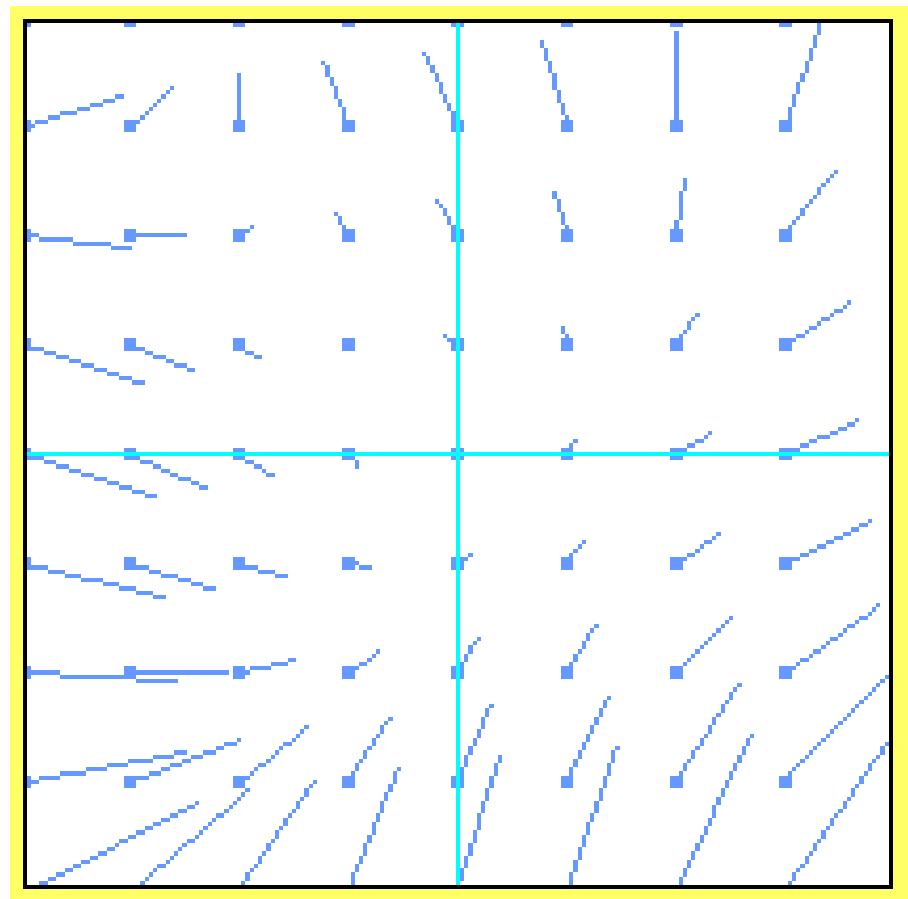

Today in Physics 217: vector derivatives

- First derivatives:
 - Gradient (∇)
 - Divergence ($\nabla \cdot$)
 - Curl ($\nabla \times$)
- Second derivatives: the Laplacian (∇^2) and its relatives
- Vector-derivative identities: relatives of the chain rule, product rule, *etc.*

Image by Eric Carlen, School of Mathematics, Georgia Institute of Technology



$$v(x, y) = (x^2 - y) \hat{x} + (x + y^2) \hat{y}$$

Differential vector calculus

df/dx provides us with information on how quickly a function of one variable, $f(x)$, changes. For instance, when the argument changes by an infinitesimal amount, from x to $x+dx$, f changes by df , given by

$$df = \left(\frac{df}{dx} \right) dx$$

In three dimensions, the function f will in general be a function of x , y , and z : $f(x, y, z)$. The change in f is equal to

$$\begin{aligned} df &= \left(\frac{\partial f}{\partial x} \right) dx + \left(\frac{\partial f}{\partial y} \right) dy + \left(\frac{\partial f}{\partial z} \right) dz \\ &= \left(\left(\frac{\partial f}{\partial x} \right) \hat{x} + \left(\frac{\partial f}{\partial y} \right) \hat{y} + \left(\frac{\partial f}{\partial z} \right) \hat{z} \right) \cdot ((dx) \hat{x} + (dy) \hat{y} + (dz) \hat{z}) \\ &\equiv \nabla f \cdot d\mathbf{l} \end{aligned}$$

Differential vector calculus (continued)

The **vector derivative operator** ∇ (“del”)

$$\nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}$$

produces a vector when it operates on scalar function $f(x,y,z)$.

∇ is a vector, as we can see from its behavior under coordinate rotations:

$$(\nabla f)' = \vec{R} \cdot \nabla f$$

but its magnitude is not a number: it is an operator.

Differential vector calculus (continued)

There are three kinds of vector derivatives, corresponding to the three kinds of multiplication possible with vectors:

- **Gradient**, the analogue of multiplication by a scalar.

$$\nabla f$$

- **Divergence**, like the scalar (dot) product.

$$\nabla \cdot \mathbf{v}$$

- **Curl**, which corresponds to the vector (cross) product.

$$\nabla \times \mathbf{v}$$

Gradient

The result of applying the vector derivative operator on a scalar function f is called the **gradient of f** :

$$\nabla f = \left(\frac{\partial f}{\partial x} \right) \hat{\mathbf{x}} + \left(\frac{\partial f}{\partial y} \right) \hat{\mathbf{y}} + \left(\frac{\partial f}{\partial z} \right) \hat{\mathbf{z}}$$

The direction of the gradient points in the direction of maximum increase of f (i.e. “uphill”), and the magnitude of the gradient gives the slope of f in the direction of maximum increase.

Divergence

The scalar product of the vector derivative operator and a vector function is called the **divergence** of the vector function:

$$\nabla \cdot v = \left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \cdot (v_x \hat{x} + v_y \hat{y} + v_z \hat{z}) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

The divergence of a vector function is a scalar.

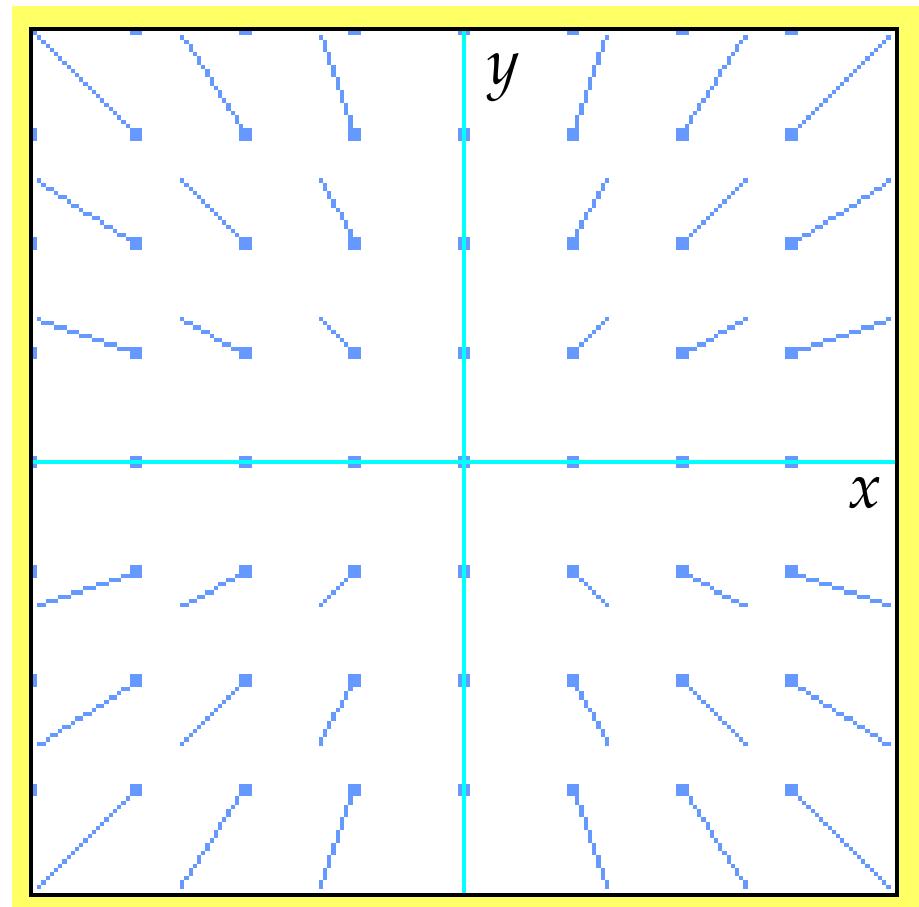
What is the divergence? If two objects following the direction specified by the vector function increase their separation, the divergence of the vector function is positive. If their separation decreases, the divergence of the vector function is negative.

A function with constant divergence

This function has

$$\nabla \cdot v = 2$$

Image by Eric Carlen
(Georgia Tech).



$$v(x, y) = x\hat{x} + y\hat{y}$$

Curl

The curl of a vector function v is

$$\nabla \times v = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{x} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{y} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{z}$$

and is, itself, a vector. (To be precise: if v is a vector function, its curl is a pseudovector function.)

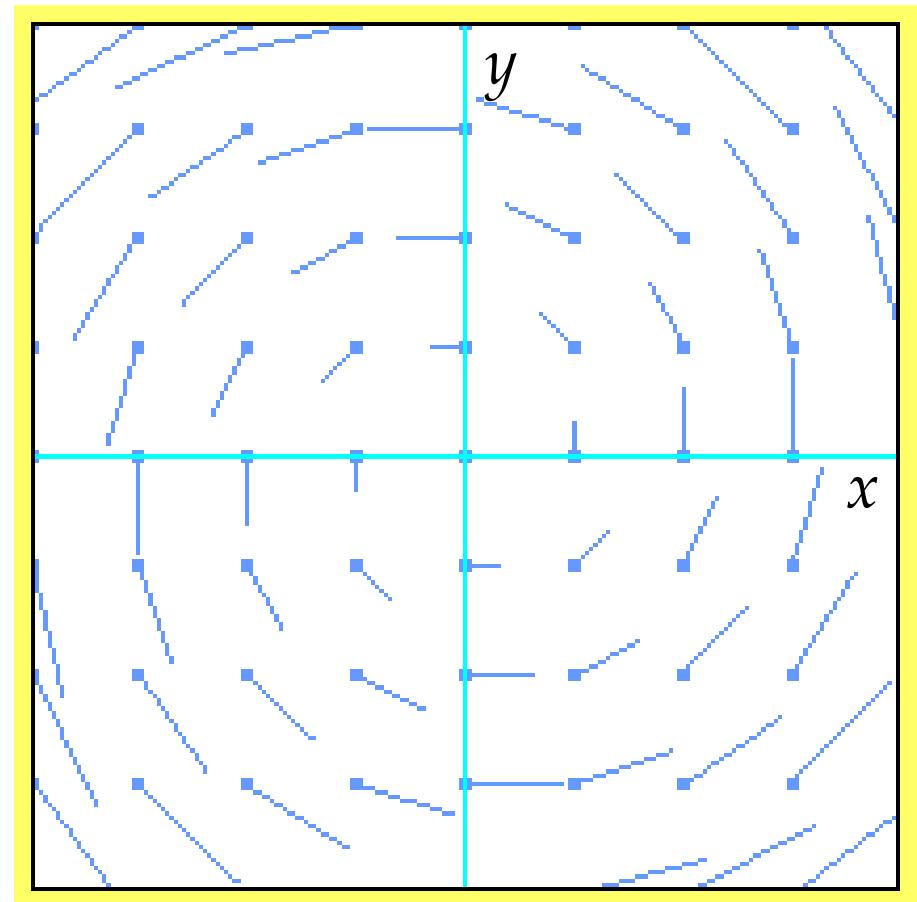
What is the curl? The curl of a vector function evaluated at a certain point is a measure of how much the vector function's direction wraps around that point. If there were nearby objects moving in the direction of the function, they would circulate about that point, if the curl were nonzero.

A function with constant curl

This function has

$$\nabla \times v = 2\hat{z}$$
.

Image by Eric Carlen
(Georgia Tech).

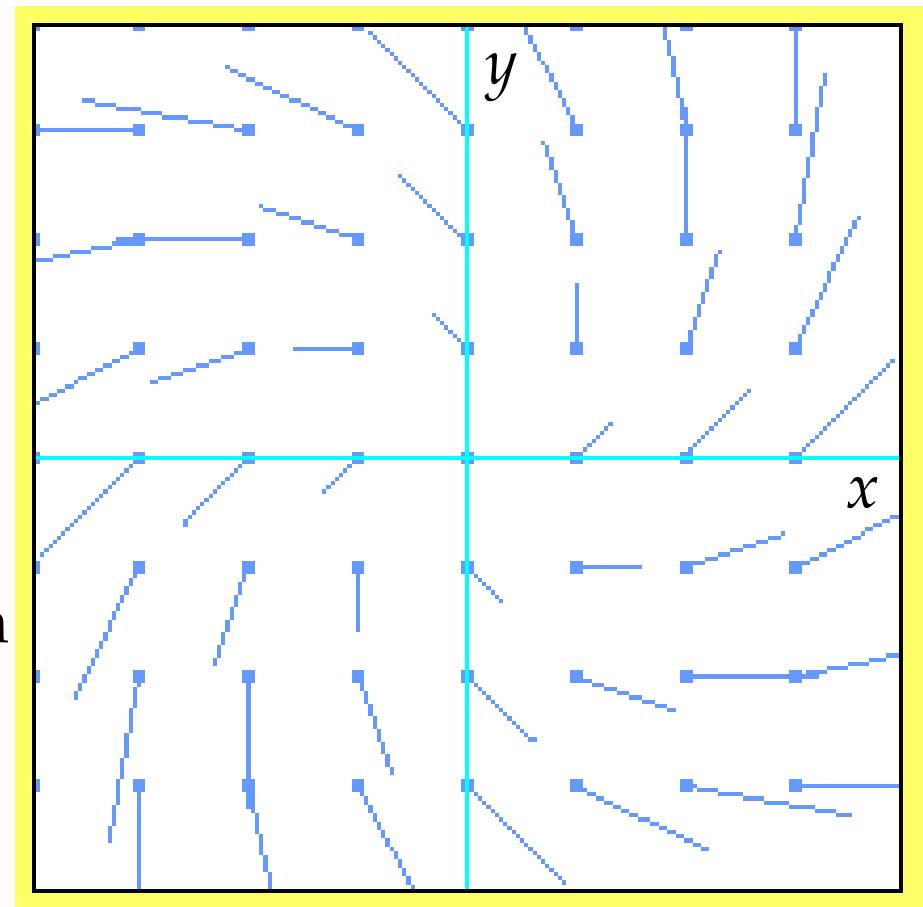


$$v(x, y) = -y\hat{x} + x\hat{y}$$

A function with constant curl and divergence

The two previous functions had nonzero divergence and zero curl, or vice versa. The sum of the two functions, shown here, has (constant) nonzero divergence *and* curl.

Image by Eric Carlen (Georgia Tech).



$$v(x, y) = (x - y)\hat{x} + (x + y)\hat{y}$$

And so on...

Here's one with nonzero, nonconstant divergence and constant curl :

$$\nabla \cdot \mathbf{v} = 2(x + y)$$

$$\nabla \times \mathbf{v} = 2\hat{z}$$

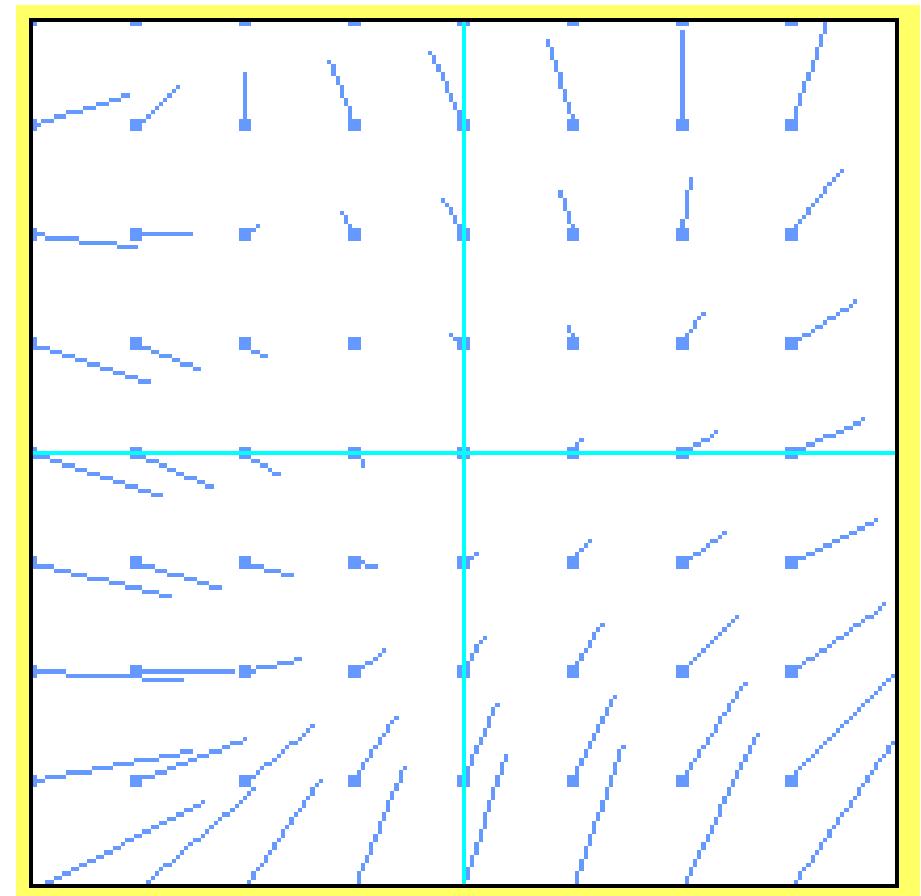


Image by Eric Carlen
(Georgia Tech).

$$\mathbf{v}(x, y) = (x^2 - y)\hat{x} + (x + y^2)\hat{y}$$

Visualization of divergence and curl

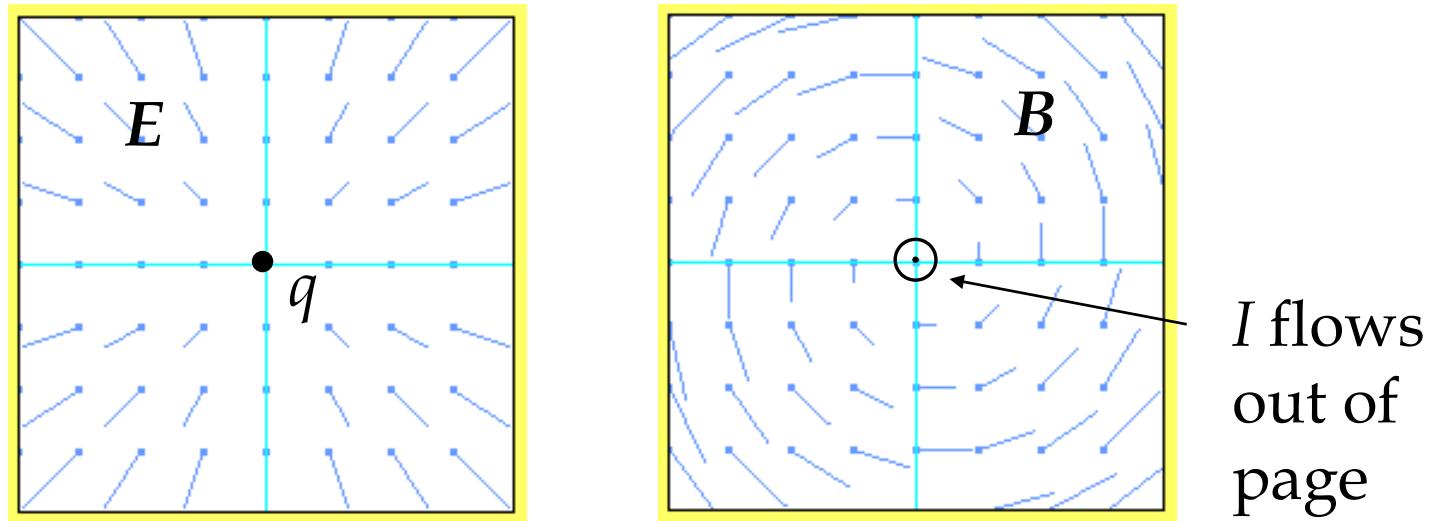
Go to this excellent Web site, from which I borrowed the figures at which we've been looking:

www.math.gatech.edu/~carlen/2507/notes/vectorCalc/

(Done by Georgia Tech math professor Eric Carlen).

Why are div and curl important in E&M?

Consider the electric field from a point charge, and the magnetic field from a constant current in a long straight wire:



Nonzero divergence of E indicates the presence of charge; nonzero curl of B indicates the presence of current. These vector derivatives point to the **sources** of the E and B fields.

Product rules for vector first derivatives

The following product rules involving the vector product will be used frequently:

$$\nabla(fg) = f \cdot \nabla g + g \cdot \nabla f$$

$$\nabla(A \cdot B) = A \times (\nabla \times B) + B \times (\nabla \times A) + (A \cdot \nabla)B + (B \cdot \nabla)A$$

$$\nabla \cdot (fA) = f(\nabla \cdot A) + A \cdot \nabla f$$

$$\nabla \cdot (A \times B) = B \cdot (\nabla \times A) - A \cdot (\nabla \times B)$$

$$\nabla \times (fA) = f(\nabla \times A) + A \times \nabla f$$

$$\nabla \times (A \times B) = (B \cdot \nabla)A - (A \cdot \nabla)B + A(\nabla \cdot B) - B(\nabla \cdot A)$$

You'll also find them on the inside front cover of Griffiths, and will prove some of them yourself in recitation.

Vector second derivatives

There are five possibilities for second derivatives involving ∇ :

$$\begin{array}{lll} \nabla \cdot (\nabla f) & \nabla \times (\nabla f) & \nabla (\nabla \cdot \mathbf{v}) \\ \nabla \cdot (\nabla \times \mathbf{v}) & \nabla \times (\nabla \times \mathbf{v}) & \end{array}$$

- The divergence of a gradient is called the **Laplacian**, denoted ∇^2 :

$$\begin{aligned} \nabla^2 f &\equiv \nabla \cdot (\nabla f) = \left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \cdot \left(\hat{x} \frac{\partial f}{\partial x} + \hat{y} \frac{\partial f}{\partial y} + \hat{z} \frac{\partial f}{\partial z} \right) \\ &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \end{aligned}$$

Soon you'll be good friends with this operator.

Vector second derivatives (continued)

- The curl of a gradient is always zero, as you'll show in this week's homework:

$$\nabla \times (\nabla f) = 0$$

- The gradient of a divergence,

$$\nabla(\nabla \cdot v)$$

appear frequently in the equations of fluid mechanics, but it never lasts long in the equations of electrodynamics.

- The divergence of a curl is always zero, as you'll also show in this week's homework:

$$\nabla \cdot (\nabla \times v) = 0$$

Vector second derivatives (continued)

- The curl of a curl of a vector function can be expressed in terms of the Laplacian and the gradient of the divergence of the vector function:

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla (\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$$

so it's not really different from the other four.
(Note, while you're at it, that the Laplacian can operate on scalar or vector functions.)