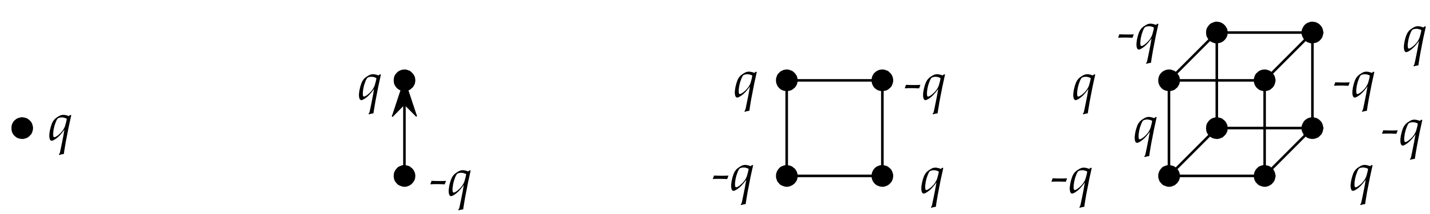
Today in Physics 217: multipole expansion

- ❑ Multipole expansions
- ❑ Electric multipoles and their moments
  - Monopole and dipole, in detail
  - Quadrupole, octupole, …
- ❑ Example use of multipole expansion as approximate solution to potential from a charge distribution (Griffiths problem 3.26)



Solving the Laplace and Poisson equations by sleight of hand

The guaranteed uniqueness of solutions has spawned several creative ways to solve the Laplace and Poisson equations for the electric potential. We will treat three of them in this class:

- **Method of images** (9 October).
  Very powerful technique for solving electrostatics problems involving charges and conductors.
- **Separation of variables** (11-18 October)
  Perhaps the most useful technique for solving partial differential equations. You'll be using it frequently in quantum mechanics too.
- **Multipole expansion** (today)
  Fermi used to say, "When in doubt, expand in a power series." This provides another fruitful way to approach problems not immediately accessible by other means.

Multipole expansions

Suppose we have a known charge distribution for which we want to know the potential or field *outside* the region where the charges are. If the distribution were symmetrical enough we could find the answer by several means:

- direct calculation using Gauss' Law;
- direct calculation Coulomb's Law;
- solution of the Laplace equation, using the charge distribution for boundary conditions.

But even when ρ is symmetrical this can be a lot of work. Moreover, it may give more precise information on the potential or field than is actually needed.

Consider instead a direct calculation combined with a series expansion…

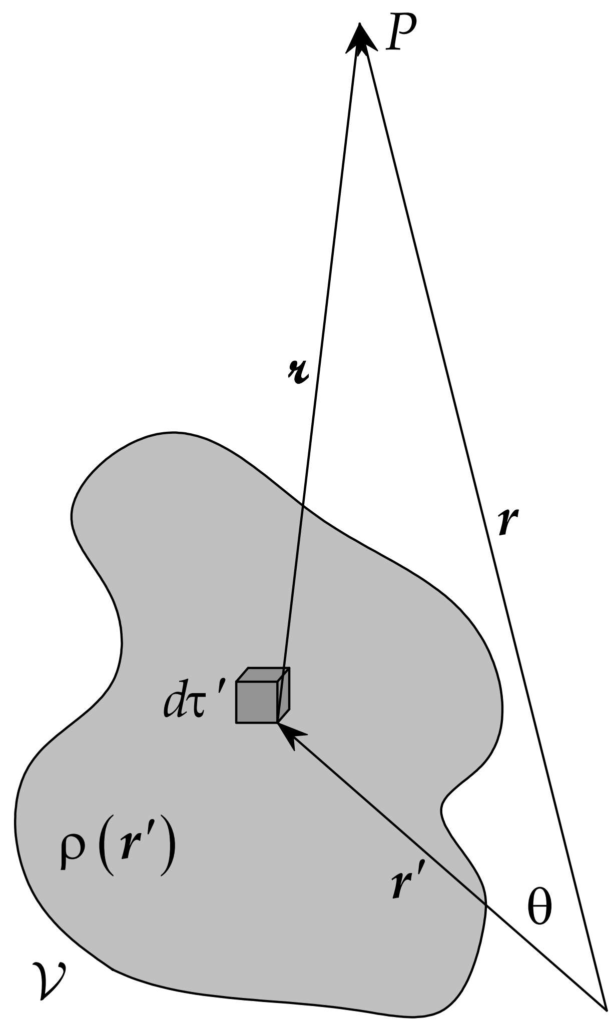
Multipole expansions (continued)

If the reference point for potential is (and can be) at infinity, then

$$V(\boldsymbol{r}) = \int_{\mathcal{V}} \frac{\rho\, d\tau'}{\boldsymbol{\mathscr{r}}} \quad ,$$

where
$$\begin{aligned}\mathscr{r}^2 &= r^2 + r'^2 - 2rr'\cos\theta \\ &= r^2\left(1 + \left[\frac{r'}{r}\right]^2 - 2\frac{r'}{r}\cos\theta\right) \\ &\equiv r^2(1+\varepsilon) \quad .\end{aligned}$$

If point P is far away from the charge distribution, then $\varepsilon \ll 1$.



Multipole expansions (continued)

So consider $\dfrac{1}{\mathscr{r}} = \dfrac{1}{r}\dfrac{1}{\sqrt{1+\varepsilon}}$. But first recall this infinite series:

$$(1+x)^s = \sum_{n=0}^{\infty} \frac{s!}{n!(s-n)!} x^n$$

$$= 1 + \frac{s}{1!}x + \frac{s(s-1)}{2!}x^2 + \frac{s(s-1)(s-2)}{3!}x^3 + \ldots$$

where $|x| < 1$ and s is any real number. (This is one form of the **binomial theorem.**) Then

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left(1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 - \frac{5}{16}\varepsilon^3 + \ldots\right)$$

Multipole expansions (continued)

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left[1 - \frac{1}{2}\frac{r'}{r}\left(\frac{r'}{r} - 2\cos\theta\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2 \boxed{\left(\frac{r'}{r} - 2\cos\theta\right)^2}\right.$$

$$\left. - \frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\theta\right)^3 + \ldots\right]$$

$$= \left(\frac{r'}{r}\right)^2 + 4\cos^2\theta - 4\frac{r'}{r}\cos\theta$$

$$= \frac{1}{r}\left[1 + \frac{r'}{r}\cos\theta - \frac{1}{2}\left(\frac{r'}{r}\right)^2 + \frac{3}{2}\left(\frac{r'}{r}\right)^2\cos^2\theta - \frac{3}{2}\left(\frac{r'}{r}\right)^3\cos\theta + \frac{3}{8}\left(\frac{r'}{r}\right)^4\right.$$

$$- \frac{5}{16}\left(\frac{r'}{r}\right)^3\left\{\left(\frac{r'}{r}\right)^3 + \frac{4r'}{r}\cos^2\theta - 4\left(\frac{r'}{r}\right)^2\cos\theta\left(\frac{r'}{r}\right)^3\right.$$

$$\left.\left. -2\left(\frac{r'}{r}\right)^2\cos\theta - 8\cos^3\theta + 8\frac{r'}{r}\cos^2\theta\right\} + \ldots\right]$$

Multipole expansions (continued)

Collect terms with the same powers of r'/r, and ignore higher powers than $\left(r'/r\right)^3$, for now:

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left[1 + \frac{r'}{r}\cos\theta + \left(\frac{r'}{r}\right)^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + \left(\frac{r'}{r}\right)^3\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right) + \ldots\right]$$

$$= \frac{1}{r}\left[P_0(\cos\theta) + \frac{r'}{r}P_1(\cos\theta) + \left(\frac{r'}{r}\right)^2 P_2(\cos\theta) + \left(\frac{r'}{r}\right)^3 P_3(\cos\theta) + \ldots\right]$$

Multipole expansions (continued)

Thus, $\dfrac{1}{\boldsymbol{\mathscr{r}}} = \dfrac{1}{r}\sum_{n=0}^{\infty}\left(\dfrac{r'}{r}\right)^{n} P_n(\cos\theta)$

$$V(\boldsymbol{r}) = \frac{1}{r}\sum_{n=0}^{\infty}\int_{\mathcal{V}} \rho(\boldsymbol{r}')\left(\frac{r'}{r}\right)^{n} P_n(\cos\theta)\,d\tau'$$

$$= \frac{1}{r}\int_{\mathcal{V}} \rho(\boldsymbol{r}')\,d\tau' + \frac{1}{r^2}\int_{\mathcal{V}} \rho(\boldsymbol{r}')\,r'\cos\theta\,d\tau' \quad \textbf{Monopole, Dipole}$$

$$+ \frac{1}{r^3}\int_{\mathcal{V}} \rho(\boldsymbol{r}')\,r'^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)d\tau' \quad \textbf{Quadrupole}$$

$$+ \frac{1}{r^4}\int_{\mathcal{V}} \rho(\boldsymbol{r}')\,r'^3\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right)d\tau' + \ldots \quad \textbf{Octupole}$$

Electric multipoles

This is a useful approximation scheme, the more useful the further away point P is from the charges within $\mathcal{V}$, because one can neglect the higher-order terms in the series after the desired accuracy is achieved.

- The monopole term:

$$V_{\text{monopole}}(\boldsymbol{r}) = \frac{1}{r}\int_{\mathcal{V}} \rho(\boldsymbol{r}')\,d\tau' = \frac{q}{r}$$

If a charge distribution has a net total charge, it will tend to look like a monopole (point charge) from large distances.

Electric multipoles

- The dipole term:

$$V(\boldsymbol{r}) = \frac{1}{r^2}\int_{\mathcal{V}} \rho(\boldsymbol{r}')\, r' \cos\theta \, d\tau' = \frac{\hat{\boldsymbol{r}}}{r^2} \cdot \int_{\mathcal{V}} \rho(\boldsymbol{r}')\, \boldsymbol{r}' d\tau' = \frac{\hat{\boldsymbol{r}}}{r^2} \cdot \boldsymbol{p} \quad ,$$

where $\boldsymbol{p} = \int_{\mathcal{V}} \rho(\boldsymbol{r}')\, \boldsymbol{r}' d\tau'$ is called the **dipole moment.**

As usual, for surface, line and point charges, we have

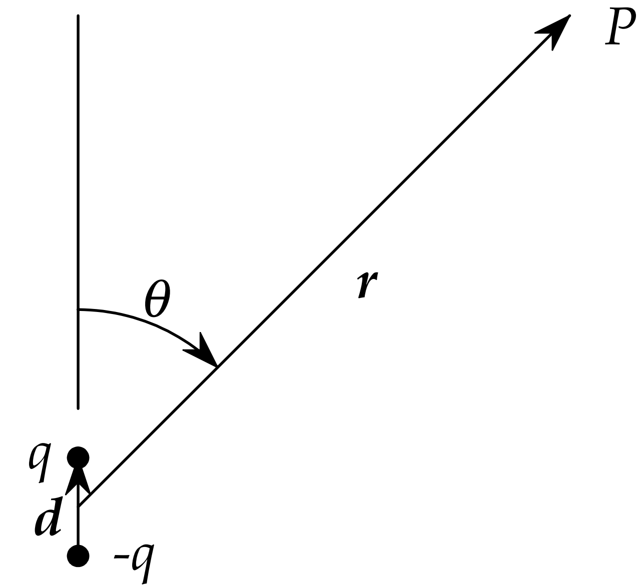
$$\boldsymbol{p} = \int_{\mathcal{S}} \sigma(\boldsymbol{r}')\, \boldsymbol{r}' da', \quad \boldsymbol{p} = \int_{\mathcal{C}} \lambda(\boldsymbol{r}')\, \boldsymbol{r}' d\ell', \quad \boldsymbol{p} = \sum_{i=1}^{n} q_i \boldsymbol{r}_i' \quad .$$

The simplest dipole has two point charges, $\pm q$, separated by a displacement vector $\boldsymbol{d}$ that points from $-q$ to $+q$.

The dipole potential and field

$$V_{\mathrm{dipole}} = \frac{qd\cos\theta}{r^2} = \frac{p\cos\theta}{r^2}$$

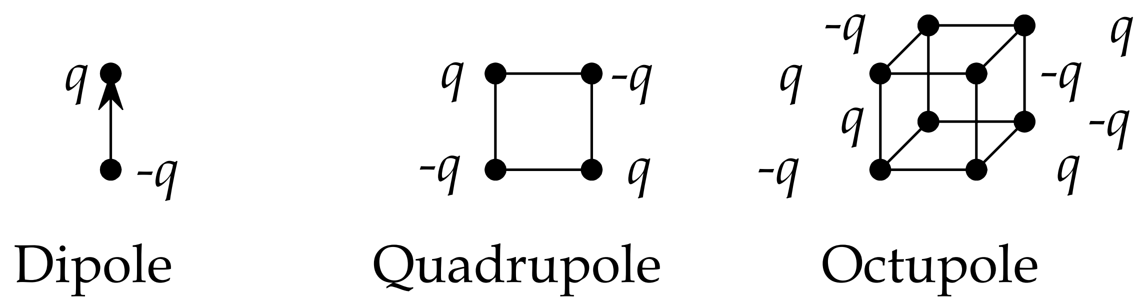
$$\begin{aligned}\boldsymbol{E}_{\mathrm{dipole}} &= -\nabla V_{\mathrm{dipole}} \\ &= -\left(\frac{\partial V_{\mathrm{dipole}}}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial V_{\mathrm{dipole}}}{\partial\theta}\hat{\boldsymbol{\theta}}\right) \\ &= \frac{2p\cos\theta}{r^3}\hat{\boldsymbol{r}} + \frac{p\sin\theta}{r^3}\hat{\boldsymbol{\theta}} \propto \frac{1}{r^3}\end{aligned}$$



Dipole moment is defined the same way in cgs and MKS. Expressions for potential and field still need a factor of $1/4\pi\varepsilon_0$ to convert from cgs to MKS.

Quadrupole, octupole,…

A simple way to envision what the higher-order multipoles “look like” is to construct them from the lower-order ones: take two of the lower-order ones, invert one, and place the two in close proximity.



The monopole moment (charge) is a scalar. The dipole moment is a vector. Higher order multipole moments are represented by higher-order tensors: the quadrupole moment is a second-rank tensor, *etc.*

Example of the use of multipole expansions

Griffiths problem 3.26: A sphere of radius R, centered at the origin, carries charge density

$$\rho\left(r,\theta\right) = k\frac{R}{r^2}\left(R-2r\right)\sin\theta\,,$$

where k is a constant and r and θ are the usual spherical coordinates. Find the approximate potential for points on the z axis, far from the sphere.

Scheme: start by calculating the monopole term. If it's not zero, then it's a good approximation to the potential, since it's larger by r/r' than the dipole term. If it is zero, move on to the dipole term. And so on…

Example (continued)

Monopole moment (charge):

$$q = \int_{\mathcal{V}} \rho(\boldsymbol{r}') \, d\tau' = kR \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \left[\frac{1}{r^2} (R - 2r) \sin\theta \right] r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= kR \int_0^{2\pi} d\phi \int_0^{\pi} \sin^2\theta \, d\theta \int_0^{R} (R - 2r) \, dr$$

$$= kR \int_0^{2\pi} d\phi \int_0^{\pi} \sin^2\theta \, d\theta \left[Rr - r^2 \right]_0^R = \boxed{0}$$

No net charge, so move on to the dipole term.

Example (continued)

Dipole moment, or lack thereof:

$$p = \int_{\mathcal{V}} r\cos\theta\rho\left(\boldsymbol{r}'\right)d\tau'$$

$$= kR\int_0^{2\pi}\int_0^{\pi}\int_0^{R} r\cos\theta\left[\frac{1}{r^2}\left(R-2r\right)\sin\theta\right]r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= kR\int_0^{2\pi} d\phi\int_0^{\pi}\sin^2\theta\,\cos\theta\,d\theta\int_0^{R}\left(Rr-2r^2\right)dr$$

$$= kR\int_0^{2\pi} d\phi\left[\frac{\sin^3\theta}{3}\right]_0^{\pi}\int_0^{R}\left(Rr-2r^2\right)dr = \boxed{0}$$

Example (continued)

Quadrupole "moment" (simple only because this is spherical):

$$Q = \int_{\mathcal{V}} r^2 \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) \rho\left(\boldsymbol{r}'\right) d\tau'$$

$$= \frac{kR}{2} \int_0^{2\pi} \int_0^{\pi} \int_0^{R} r^2 \left(3\cos^2 \theta - 1 \right) \left[\frac{1}{r^2} \left(R - 2r \right) \sin\theta \right] r^2 \sin\theta \, dr d\theta \, d\phi$$

$$= \frac{kR}{2} \int_0^{2\pi} d\phi \int_0^{\pi} \left(3\cos^2 \theta - 1 \right) \sin^2 \theta \, d\theta \int_0^{R} \left(Rr^2 - 2r^3 \right) dr$$

$$= \frac{kR}{2} \left(2\pi \right) \left(-\frac{\pi}{8} \right) \left(\frac{R^4}{3} - \frac{R^4}{2} \right) = \boxed{\frac{k\pi^2 R^5}{48} \neq 0}$$

Example (continued)

Thus, for a point way up the z axis,

$$V(\boldsymbol{r}) \cong \frac{1}{z^3} Q = \boxed{\frac{k\pi^2 R^5}{48z^3}} \qquad \left(\times \frac{1}{4\pi\varepsilon_0} \text{ for the MKS answer} \right)$$