Today in Physics 217: electric potential

- Finish Friday’s discussion of the field from a uniformly-charged sphere, and the gravitational analogue of Gauss’ Law.
- Electric potential
- Example: a field and its potential
- Poisson’s and Laplace’s equations

Electric potential

Because $\nabla \times \boldsymbol{E} = 0$ in electrostatics, we can express $\boldsymbol{E}$ as the gradient of a scalar function:

$$\boldsymbol{E} = -\nabla V$$

where of course V is called the electric scalar potential. By the gradient theorem, we can write

$$-\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{E} \cdot d\boldsymbol{l} = \int_{\boldsymbol{a}}^{\boldsymbol{b}} \left(\nabla V\right) \cdot d\boldsymbol{l} = V\left(b\right) - V\left(a\right)$$

Suppose we have agreed upon a standard reference point, $\mathcal{O}$; then

$$-\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{E} \cdot d\boldsymbol{l} = \int_{\mathcal{O}}^{\boldsymbol{b}} \left(\nabla V\right) \cdot dl + \int_{\boldsymbol{a}}^{\mathcal{O}} \left(\nabla V\right) \cdot dl$$

Electric potential (continued)

This leads us to an integral definition of V:

$$V(\mathcal{P}) = -\int_{\mathcal{O}}^{\mathcal{P}} \boldsymbol{E} \cdot d\boldsymbol{l}$$

Properties of the electric potential:

- **Arbitrariness**. An arbitrary constant can be added to the potential without changing the field (which, after all, is the fundamental quantity). To each constant corresponds a potential reference point. Thus there is always a large selection of appropriate reference points in any electrostatics problem.

Electric potential (continued)

- **Convention**: take $\mathcal{O}$ to lie at infinity, unless the charge distribution itself extends to infinity. Just remember that you can actually put the reference point anywhere that doesn't lead to an infinite result for the potential; sometimes you will find reference points not at infinity that will be more convenient for your calculation.
- **Significance**. The magnitude of the electric potential therefore has no physical significance; only *differences* in potential do.
- **Superposition**. The electric potential superposes. If $\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2 + \ldots,$ then

$$V = -\int_{\mathcal{O}}^{\mathcal{P}} \boldsymbol{E}_1 \cdot d\boldsymbol{l} - \int_{\mathcal{O}}^{\mathcal{P}} \boldsymbol{E}_2 \cdot d\boldsymbol{l} - \ldots = V_1 + V_2 + \ldots$$

Electric potential (continued)

Thus at a given point $\mathcal{P}$, one could compute Vs due to different parts of a charge distribution, and add the results to get the real thing. This may seem trivial until we find, later this week, that the closely-related electrostatic potential energy does not superpose.

From point charges and superposition we can induce the potential from a continuous charge distribution:

$$V_i = -\int_{\mathcal{O}}^{\mathcal{P}} \boldsymbol{E} \cdot d\boldsymbol{l} \xrightarrow[\mathcal{O} \to \infty]{} -\int_{\infty}^{r} E\,dr' = \frac{Q_i}{r}$$

$$V(\mathcal{P}) = \sum_{i=1}^{N} \frac{Q_i}{\mathscr{r}_i} \xrightarrow[N \to \infty]{} \int_{\mathcal{O}}^{\mathcal{P}} \frac{\rho(\boldsymbol{r}')\,d\tau'}{\mathscr{r}}$$

Electric potential (continued)

- **Units**. In CGS, it's the statvolt:

$$\text{statvolt} = \frac{\text{dyne cm}}{\text{esu}} = \frac{\text{erg}}{\text{esu}}$$

In MKS, it's the volt:

$$\text{volt} = \frac{\text{Nt m}}{\text{coul}} = \frac{\text{joule}}{\text{coul}}$$

- **Correspondence**: 1 statvolt $\leftrightarrow$ 299.792458 volts

Electric potential (continued)

- **What it's good for**. It's often easier to calculate V, and take its gradient to find ***E***, than to calculate ***E*** directly. Reasons:
  - V is a scalar; no vector addition to get it.
  - There are many situations in nature in which V can be regarded as constant over a region in space near where one would like to know ***E***. The solution of V for space between the constant-V ("equipotential") locations and the reference point – the process of which is called a boundary-value problem – can be shown to be unique. Finding V by boundary-value solution, and then calculating ***E***, is in these cases usually much easier than calculating ***E*** directly.

Example

Griffiths problem 2.20

One of these is an impossible electrostatic field. Which one?

$$\text{a. } \boldsymbol{E} = k\left[(xy)\hat{\boldsymbol{x}} + (2yz)\hat{\boldsymbol{y}} + (3xz)\hat{\boldsymbol{z}}\right]$$

$$\text{b. } \boldsymbol{E} = k\left[\left(y^2\right)\hat{\boldsymbol{x}} + \left(2xy + z^2\right)\hat{\boldsymbol{y}} + (2yz)\hat{\boldsymbol{z}}\right]$$

Here, k is a constant with the appropriate units. For the *possible* one, find the potential, using the origin as your reference point. Check your answer by computing $-\nabla V$.

First take the curl of each function:

Example (continued)

a. $$\nabla \times \boldsymbol{E} = k\left(\frac{\partial}{\partial y}(3xz) - \frac{\partial}{\partial z}(2yz)\right)\hat{\boldsymbol{x}} + k\left(\frac{\partial}{\partial z}(xy) - \frac{\partial}{\partial x}(3xz)\right)\hat{\boldsymbol{y}}$$
$$+ k\left(\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial y}(xy)\right)\hat{\boldsymbol{z}} = k\left(-2y\hat{\boldsymbol{x}} - 3z\hat{\boldsymbol{y}} - x\hat{\boldsymbol{z}}\right) \neq 0$$

b. $$\nabla \times \boldsymbol{E} = k\left(\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}\left(2xy + z^2\right)\right)\hat{\boldsymbol{x}} + k\left(\frac{\partial}{\partial z}\left(y^2\right) - \frac{\partial}{\partial x}(2yz)\right)\hat{\boldsymbol{y}}$$
$$+ k\left(\frac{\partial}{\partial x}\left(2xy + z^2\right) - \frac{\partial}{\partial y}\left(y^2\right)\right)\hat{\boldsymbol{z}} = 0$$

The first one can’t, but the second one can.

Example (continued)

Integrate $\boldsymbol{E}$ to get V: choose path for convenience since the result is path-independent.

$$(0,0,0)\rightarrow(x,0,0)\rightarrow(x,y,0)\rightarrow(x,y,z)$$

$$\boldsymbol{E}\cdot d\boldsymbol{l}=ky^2dx=0$$

$$\int_{(0,0,0)}^{(x,0,0)}\boldsymbol{E}\cdot d\boldsymbol{l}=0$$

$$\boldsymbol{E}\cdot d\boldsymbol{l}=k\left(2xy+z^2\right)dy=2kxydy$$

$$\int_{(x,0,0)}^{(x,y,0)}\boldsymbol{E}\cdot d\boldsymbol{l}=\int_0^y 2kxydy=kxy^2$$

$$\boldsymbol{E}\cdot d\boldsymbol{l}=2k\left(yz\right)dz$$

$$\int_{(x,y,0)}^{(x,y,z)}\boldsymbol{E}\cdot d\boldsymbol{l}=\int_0^z 2k\left(yz\right)dz=kyz^2$$

Example (continued)

Thus

$$V(x,y,z) = 0 - kxy^2 - kyz^2 = -k\left(xy^2 + yz^2\right)$$

Check:

$$-\nabla V = -\frac{\partial V}{\partial x}\hat{\boldsymbol{x}} - \frac{\partial V}{\partial y}\hat{\boldsymbol{y}} - \frac{\partial V}{\partial z}\hat{\boldsymbol{z}} = k\left(y^2\hat{\boldsymbol{x}} + \left(2xy + z^2\right)\hat{\boldsymbol{y}} + (2yz)\hat{\boldsymbol{z}}\right)$$

$$= \boldsymbol{E}$$

Differential equations for the electric potential

To get V without referring first to $\boldsymbol{E}$:

$$\nabla \cdot \boldsymbol{E} = 4\pi\rho$$

$$\nabla \cdot \left(-\nabla V\right) =$$

$$\boxed{\nabla^2 V = -4\pi\rho}$$ **Poisson's equation**

In regions where there are no electric charges,

$$\boxed{\nabla^2 V = 0}$$ **Laplace's equation**

These equations, plus boundary conditions, provide the boundary-value-problem way to calculate V.