Physics 237, Spring 2008
Final Exam
Wednesday, May 7, 2008

- Answer all questions, using the paper provided. *Begin each problem on a separate sheet.* The pages will be separated for grading, so clearly label the problem number and put your name on EACH PAGE.
- One 8.5 by 11 inch cheat sheet (front and back) allowed. No other books, calculators or notes allowed.
- SHOW ALL WORK.
- Potentially useful information:

$$d^3x = r^2 dr \sin\theta \, d\theta \, d\phi; \quad 0 \le \theta \le \pi; \;\; 0 \le \phi \le 2\pi$$

$$\int_0^{+\infty} e^{-ax^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} \qquad \int_0^{+\infty} x^2\, e^{-ax^2}\, dx = \frac{1}{4}\sqrt{\frac{\pi}{a^3}} \qquad \int_0^{+\infty} x\, e^{-ax^2}\, dx = \frac{1}{2a}$$

$$\int x\, e^{ax}\, dx = \frac{e^{ax}}{a^2}(ax-1) \qquad \int x^2\, e^{ax}\, dx = \frac{x^2\, e^{ax}}{a} - \frac{2}{a}\int x\, e^{ax}\, dx$$

$$\int (\sin^2 ax)\, dx = \frac{x}{2} - \frac{1}{4a}\sin 2ax \qquad \int (\cos^2 ax)\, dx = \frac{x}{2} + \frac{1}{4a}\sin 2ax$$

$$\int (\sin ax)(\cos ax)\, dx = \frac{1}{2a}\sin^2 ax \qquad \int (\sin^2 ax)(\cos^2 ax)\, dx = \frac{x}{8} - \frac{1}{32a}\sin 4ax$$

$$\int (\sin ax)(\cos^m ax)\, dx = -\frac{\cos^{m+1} ax}{(m+1)a} \qquad \int (\sin^m ax)(\cos ax)\, dx = \frac{\sin^{m+1} ax}{(m+1)a}$$

$$L_x = i\hbar(\sin\phi\; \partial/\partial\theta + \cot\theta\cos\phi\; \partial/\partial\phi) \qquad L_y = i\hbar(-\cos\phi\; \partial/\partial\theta + \cot\theta\sin\phi\; \partial/\partial\phi)$$

$$L_z = -i\hbar\, \partial/\partial\phi$$

$$\sin 2x = 2\sin x\;\cos x \qquad \cos 2x = 2\cos^2 x - 1$$

1. (20 points total) *Short answer.*

   (a) (12 points) What physical requirement (i.e. constraint on the wave function) leads to quantization of the following?

      i. Energy levels in the 1-D simple harmonic oscillator.

      ii. Energy levels in the hydrogen atom.

      iii. Total orbital angular momentum in the hydrogen atom.

      iv. z component of orbital angular momentum in the hydrogen atom.

   (b) (4 points) Give an example of experimental evidence for the existence of spin.

   (c) (4 points) Describe (in words) what gives rise to the spin-orbit interaction that leads to the fine structure in atomic spectra.

2. (25 points total) Consider the hydrogen atom and its eigenstates, omitting effects due to fine structure.

   (a) (15 points) For the hydrogen eigenstate ψ_{210}, give the following (you do not necessarily have to do an explicit calculation, but if you do not, give a reason for your answer):

      i. expectation value of the energy

      ii. expectation value of the total orbital angular momentum

      iii. expectation value of the z component of orbital angular momentum

      iv. expectation value of the x component of orbital angular momentum

      v. possible values of total angular momentum.

   (b) (10 points) Selection rules govern the transitions between hydrogen states and specify which transitions are and are not allowed. Electric dipole transition selection rules involve the electric dipole moment operator

$$\vec{p} = -e\vec{r}$$

   where e is the electron charge and $\vec{r}$ is the electron's position with respect to the proton. These transitions are not allowed if the integral of this operator between the initial and final states is equal to zero. Show explicitly that this forbids transitions between states with $m_l = -1$ and $m_l = +1$, i.e. show

$$\int \psi^*_{nl-1}\,(\vec{p})\,\psi_{n'\,l'\,+1}\,d^3x = 0$$

   for the z component of $\vec{p}$. Recall $z = r\cos\theta$, $x = r\sin\theta\cos\phi$, and $y = r\sin\theta\sin\phi$. (Hint: Concentrate on the ϕ dependence.)

3. (10 points total) Consider the one-dimensional potential $V(x) = C|x|$:

(a) (2 points) Does this potential have bound states only, free (continuum) states only, or both?

(b) (8 points) Sketch the wave functions and probability densities for the ground state and first excited state of this potential. Label clearly which is which.

4. (10 points total) If a hydrogen atom is placed in a magnetic field that is very strong compared to its internal field, its orbital and spin magnetic dipole moments precess independently about the external field, and its energy depends on the quantum numbers m_l and m_s which specify their components along the external field direction.

(a) (5 points) What are the atom's orbital and spin magnetic dipole moments, in terms of the quantum numbers m_l and m_s, and whatever else they depend on?

(b) (5 points) Evaluate the splitting of the energy levels according to the values of m_l and m_s.

5. (15 points total)

(a) (11 points) Consider the simple harmonic oscillator, with potential $V(x) = (C/2)x^2$. Estimate the ground state energy based on an argument from the uncertainty principle.

(b) (2 points) Using the energy for the simple harmonic oscillator that we derived in class, find the ratio of the ground state energies for a muon to that of an electron.

(c) (2 points) Answer (b) for a particle in a box (infinite square well).

6. (20 points total)

Consider the isotropic simple harmonic oscillator (SHO) in 3 dimensions, which is described by the potential $V(\vec{r}) = (C/2)(x^2 + y^2 + z^2) = (C/2)r^2$. We can solve this system two ways: by separating it in Cartesian coordinates or in spherical coordinates. In each case we get a (different) set of three quantum numbers corresponding to the three degrees of freedom in the problem.

(a) (12 points) By performing the separation in Cartesian coordinates, and using your knowledge of the one-dimensional SHO, find the total energy of the 3D SHO. To do this:

i. Write down the time-independent Schroedinger equation and perform the separation, using

$$\psi(x, y, z) = X(x)Y(y)Z(z).$$

(Hint: You will want to introduce constants E_x, E_y, and E_z that add up to the total energy E.)

ii. Show that each of the functions X, Y, and Z is described by the Schroedinger equation for the *one*-dimensional SHO.

iii. Without solving the equations explicitly, use your knowledge of the 1D solution to introduce three quantum numbers, then express E_x, E_y, E_z in terms of them, and sum the energies to get an expression for E in terms of your three quantum numbers (and anything else it might depend on).

(b) (8 points) Now forget about Cartesian coordinates and think in terms of spherical coordinates. By analogy with the solution to the Schroedinger equation for the hydrogen atom, make an argument for what *two* of the quantum numbers for the isotropic SHO must be for the solution in spherical coordinates. What is the ϕ dependence of the solution? You do *not* need to perform the separation of variables explicitly, but you must be very clear in your argument about where the analogy holds. What is the relevant similarity of the SHO and hydrogen atom potentials? What does that imply is the same in the solutions for the two potentials? What is not the same?

P237 Final exam sol'ns Spring '08

1 a) Physical requirement/condition on wave fn that leads to quantization

i. 1-D S.H.O. energy levels: ψ finite, i.e. can't have $\psi \to \infty$ as $|x| \to \infty$

ii. H energy levels: same as i

iii. Total L in H atom: finiteness, i.e. ψ finite as $\theta \to 0, \pi$

iv. L_z in H atom ψ must be single-valued

b) Experimental evidence for spin

- splitting of beam into two components in Stern-Gerlach
- spin-orbit coupling or fine structure splitting in H

c) Spin-orbit interaction: In the electron's frame of reference, the nucleus is a moving charge which generates a magnetic field. The electron's spin magnetic moment interacts with the field. The nuclear motion the electron sees, and hence the B field, depends on the electron's orbital angular momentum.

(2,1)

[2] H atom eigenstates, omitting fine structure

a) ψ_{210} expectation values $\quad n=2,\ l=1,\ m_l=0$

i) Energy: ψ_{210} is energy eigenstate

$$E_n = \frac{-\mu e^4}{(4\pi\epsilon_0)^2 2\hbar^2}\frac{1}{n^2} = \frac{-\mu e^4}{(4\pi\epsilon_0)^2 2\hbar^2}\frac{1}{4} = -\frac{13.6\ eV}{4}$$

ii) Total orbital angular momentum: also an eigenstate

$$L = \hbar\sqrt{l(l+1)} = \hbar\sqrt{2} \text{ for } l=1$$

iii) L_z: also an eigenstate

$$L_z = \hbar m_l = 0$$

iv) L_x: Not an eigenstate, so we have to compute explicitly.

From front of test, $L_x = i\hbar\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)$

$$\psi_{210} = N\frac{r}{a_0}e^{-r/2a_0}\cos\theta \qquad N = \frac{1}{4\sqrt{\pi a_0^3}}$$

$$\frac{\partial}{\partial\theta}\psi_{210} = -N\frac{r}{a_0}e^{-r/2a_0}\sin\theta$$

$$\frac{\partial}{\partial\varphi}\psi_{210} = 0$$

$$\Rightarrow \overline{L_x} = \int \psi^*_{210} L_x \psi_{210}\, d^3x$$

$$= N^2 i\hbar\int_0^\infty\left(\frac{r}{a_0}\right)^2 r^2 dr \underbrace{\int_0^\pi \sin^2\theta\cos\theta\, d\theta}_{\frac{\sin^3\theta}{3}\Big|_0^\pi = 0-0=0} \underbrace{\int_0^{2\pi} d\varphi\sin\varphi}_{-\cos\varphi\big|_0^{2\pi} = -1-(-1) = 0}$$

r/ front of test

$$\Rightarrow \overline{L_x} = 0$$

2, cont

a, cont

v) Possible total angular momentum

$$j = l + \frac{1}{2},\ l - \frac{1}{2} = \frac{3}{2},\ \frac{1}{2}$$

$$\text{so } J = \hbar\sqrt{j(j+1)} = \frac{\hbar}{2}\sqrt{15}\ ,\ \frac{\hbar}{2}\sqrt{3}$$

b) Electric dipole transition

show $\int \psi^*_{n,l,-1}\, P_z\, \psi_{n',l',-1}\, d^3x = 0$

where $\vec{p} = -e\vec{r}$

This is true for any n, l, n', l' so we only need the φ dependent part of the wave fn.

$$\psi_{nl-1} = R_{nl}\,\Theta_{l,-1}\, e^{-i\varphi} \Rightarrow \psi^*_{nl-1} = R^*_{nl}\,\Theta^*_{l,-1}\, e^{+i\varphi}$$

$$\psi_{n'l'+1} = R_{n'l'}\,\Theta_{l'+1}\, e^{i\varphi}$$

and $p_z = -er\cos\theta$

So integral is

$$\int \psi^*_{nl-1}\, P_z\, \psi_{nl'+1}\, d^3x$$

$$= \int R^*_{nl}(r)\,\Theta^*_{l-1}(\theta)\, e^{+i\varphi}(-er\cos\theta) R_{n'l'}(r)\,\Theta_{l+1}(\varphi)\, e^{+i\varphi} * r^2 dr \sin\theta\, d\theta\, d\varphi$$

$$= -e\int R^*_{nl} R_{n'l'} r^3 dr \int \Theta^*_{l-1}\,\Theta_{l+1} \cos\theta \sin\theta\, d\theta\, \underbrace{\int_0^{2\pi} e^{2i\varphi}\, d\varphi}_{\text{Let} \equiv I_\varphi}$$

2b, cont

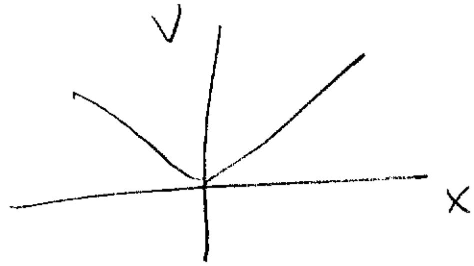
It suffices to show that $I_\varphi = 0$

$$I_\varphi = \int_0^{2\pi} e^{2i\varphi} d\varphi = \frac{1}{2i} e^{2i\varphi}\Big|_0^{2\pi} = \frac{1}{2i}(1-1) = 0$$

$\therefore$ entire integral

$$\int \psi^*_{n\ell-1} \, p_z \, \psi_{n'\ell'+1} \, d^3x = 0$$

$\boxed{3}$ $V(x) = C|x|$

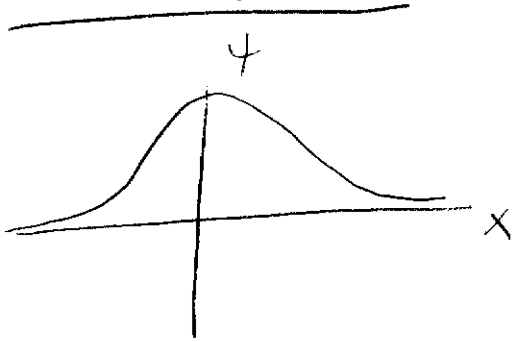


a) This potential has bound states only

b) Wave fns + probability densities

Ground state:

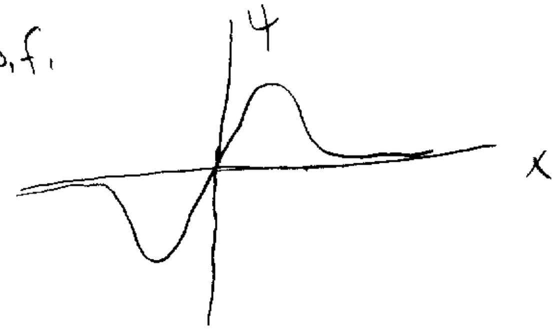
w.f.



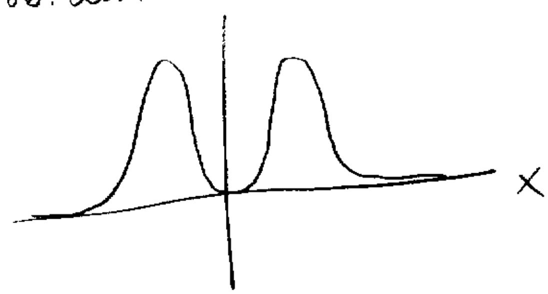
prob. dens $= |\psi|^2$

1^{st} excited state:

w.f.



prob. dens $|\psi|^2$



$\boxed{4}$ H atom in strong external field

a) Orbital & spin magnetic dipole moments

orb: $\vec{\mu}_\ell = -\dfrac{g_\ell \mu_b}{\hbar} \vec{L}$ & $L_z = \hbar m_\ell$

$g_\ell = 1$, $\mu_b = \dfrac{e\hbar}{2m}$

$= -\dfrac{e}{2m} \vec{L}$

spin: $\vec{\mu}_s = -\dfrac{g_s \mu_b}{\hbar} \vec{S}$ and $S_z = \hbar m_s$

$g_s = 2$, same μ_b

$= -\dfrac{e}{m} \vec{S}$

b) Energy splitting : $\Delta E = -\vec{\mu} \cdot \vec{B}$

orb : $\Delta E = -\vec{\mu}_\ell \cdot \vec{B} = -\mu_{\ell,z} B = +\dfrac{e}{2m} L_z B$

$= \dfrac{e}{2m} \hbar B m_\ell = +g_\ell \mu_b m_\ell B$

spin $\Delta E = -\vec{\mu}_s \cdot \vec{B} = -\mu_{s,z} B$

$= +\dfrac{e}{m} \hbar B m_s = +g_s \mu_b m_s B$

where z direction is taken to be direction of $\vec{B}$

[5] a) SHO $V = \frac{c}{2} x^2$

Estimate g.s. energy based on uncertainty princ.

$$E = \frac{p^2}{2m} + \frac{c}{2} x^2$$

Uncertainty principle $\Delta p \, \Delta x \gtrsim \frac{\hbar}{2}$

In ground state, take unc product near min:

$$\Delta p \, \Delta x \sim \hbar$$

Also, take $\Delta p \sim p$, $\Delta x \sim x \Rightarrow p \simeq \frac{\hbar}{x}$

$$\Rightarrow E = \frac{\hbar^2}{2 m x^2} + \frac{c}{2} x^2$$

Minimize for g.s.: $\frac{dE}{dx} = 0$

$$\frac{dE}{dx} = -\frac{\hbar^2}{m x^3} + c x = 0 \Rightarrow c x^4 = \frac{\hbar^2}{m}$$

$$\text{or } x = \left(\frac{\hbar^2}{c m}\right)^{1/4}$$

Plugging into E,

$$E = \frac{\hbar^2}{2m} \sqrt{\frac{cm}{\hbar^2}} + \frac{c}{2} \sqrt{\frac{\hbar^2}{cm}} = \frac{\hbar}{2} \sqrt{\frac{c}{m}} = \frac{h\nu}{2}, \quad \nu = \frac{1}{2\pi} \sqrt{\frac{c}{m}}$$

$$\Rightarrow E \sim \frac{\hbar}{2} \sqrt{\frac{c}{m}}$$ which is the actual g.s. energy

5. cont

b) Ratio E(muon)/E(electron) for SHO g.s :

$$\frac{E_\mu}{E_e} = \sqrt{\frac{m_e}{m_\mu}} \simeq \frac{1}{\sqrt{200}}$$

c) Same as b for particle in box

$$E_{g.s} = \frac{\pi^2 \hbar^2}{2ma^2} \quad \text{so} \quad \frac{E_\mu}{E_e} = \frac{m_e}{m_\mu} \simeq \frac{1}{200}$$

6 Isotropic SHO

a) Cartesian coords

$$V = \frac{c}{2}(x^2+y^2+z^2)$$

i) Separation of SE

$$-\frac{\hbar \nabla^2}{2m}\psi + \frac{c}{2}(x^2+y^2+z^2)\psi = E\psi$$

& let $\psi = X(x)\,Y(y)\,Z(z)$

& with $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, substituting ⇒

$$-\frac{\hbar^2}{2m}\left[Y Z \frac{d^2 X}{dx^2} + X Z \frac{d^2 Y}{dy^2} + X Y \frac{d^2 Z}{dz^2}\right]$$

$$+ \frac{c}{2}(x^2+y^2+z^2) X Y Z = E X Y Z$$

Now divide by $X Y Z$ and rearrange

$$\underbrace{\left(-\frac{\hbar^2}{2m}\frac{1}{X}\frac{d^2X}{dx^2} + \frac{c}{2}x^2\right)}_{\text{fn of } x \text{ only}} + \underbrace{\left(-\frac{\hbar^2}{2m}\frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{c}{2}y^2\right)}_{\text{fn of } y \text{ only}}$$

$$+ \underbrace{\left(-\frac{\hbar^2}{2m}\frac{1}{Z}\frac{d^2Z}{dz^2} + \frac{c}{2}z^2\right)}_{\text{fn of } z \text{ only}} = E$$

6a, cont

Now the terms in parentheses are, respectively, functions of x only, y only, and z only. The equation must be true for all x, y, and z, so each term in parentheses must be separately equal to a constant. Call the constants E_x, E_y and E_z so

We must have

$$-\frac{\hbar^2}{2m}\frac{1}{X}\frac{d^2X}{dx^2}+\frac{C}{2}x^2=E_x \qquad (*)$$

$$-\frac{\hbar^2}{2m}\frac{1}{Y}\frac{d^2Y}{dy^2}+\frac{C}{2}y^2=E_y$$

$$-\frac{\hbar^2}{2m}\frac{1}{Z_1}\frac{d^2Z_1}{dz^2}+\frac{C}{2}z^2=E_z$$

and $E_x+E_y+E_z=E$

ii) Show that each of X, Y + Z_1 satisfies 1D SHO eq'n. Take eq'n (*) for X and multiply by X ⇒

$$-\frac{\hbar^2}{2m}\frac{d^2X}{dx^2}+\frac{C}{2}x^2X=E_xX$$

which is the eq'n for the 1D SHO

6a, cont

ii, cont

and similarly for Y and z

iii) Use knowledge of 1D sol'n to introduce quantum no's & find E_x, E_y, E_z, & E.

We know for 1-D SHO

$$E_{1D} = h\nu(n+\tfrac{1}{2}) \quad \text{where } \nu = \frac{1}{2\pi}\sqrt{\frac{c}{m}}$$

and $n = 0, 1, \ldots$

So let n_x be quantum no. for X, n_y for Y, and n_z for Z. Note ν is the same for X, Y, and z because c and m are the same for them. So

$$E_x = h\nu(n_x + \tfrac{1}{2})$$

$$E_y = h\nu(n_y + \tfrac{1}{2})$$

$$E_z = h\nu(n_z + \tfrac{1}{2})$$

and $E = E_x + E_y + E_z = h\nu(n_x + n_y + n_z + \tfrac{3}{2})$

b) Spherical coords

$$V = \frac{1}{2} c r^2$$

This potential depends only on r, and is independent of θ and φ. By analogy w/ the solution for the hydrogen atom, we know the isotropic SHO will have the same angular solns as the H atom, w/ the quantum nos

l with $L^2 = \hbar^2 l(l+1)$

m_l $\quad L_z = \hbar m_l$

As in the H atom, the φ dependence will be

$$e^{i m_l \varphi}$$

The relevant similarity of the two potentials is that $V = V(r)$ only, indep. of $\theta + \varphi$, which implies the same angular dependence in the isotropic SHO and H atom. What is not the same is the r dependence, and the resulting energies.