Superposition and Compatible Observables

Liboff Chap. 5

The Superposition Principle

We can combine the ideas of the previous chapter
- expansion of functions in terms of basis functions
- eigenfunctions of Hermitian operators ⟺ physical observables form orthogonal sets

into the superposition principle:

$\boxed{\text{An arbitrary state } \psi \text{ can be represented as a superposition of eigenstates of a physical observable.}}$

This provides a powerful way of extracting information about the system.

To see what this means, let's back up and consider an ensemble, or collection, of identical replicas of a particle in a 1D box. Identical means same size box, same mass ~~size~~ particle, and same wave function.

We take all of the boxes to have initially

same $\psi(x,0)$, not necessarily an energy eigenstate!

then since they all satisfy the same time-dep. Schr. eq'n, they'll all have the

same $\psi(x,t)$

at some later time t. See top two lines on p. 4.2

4.2

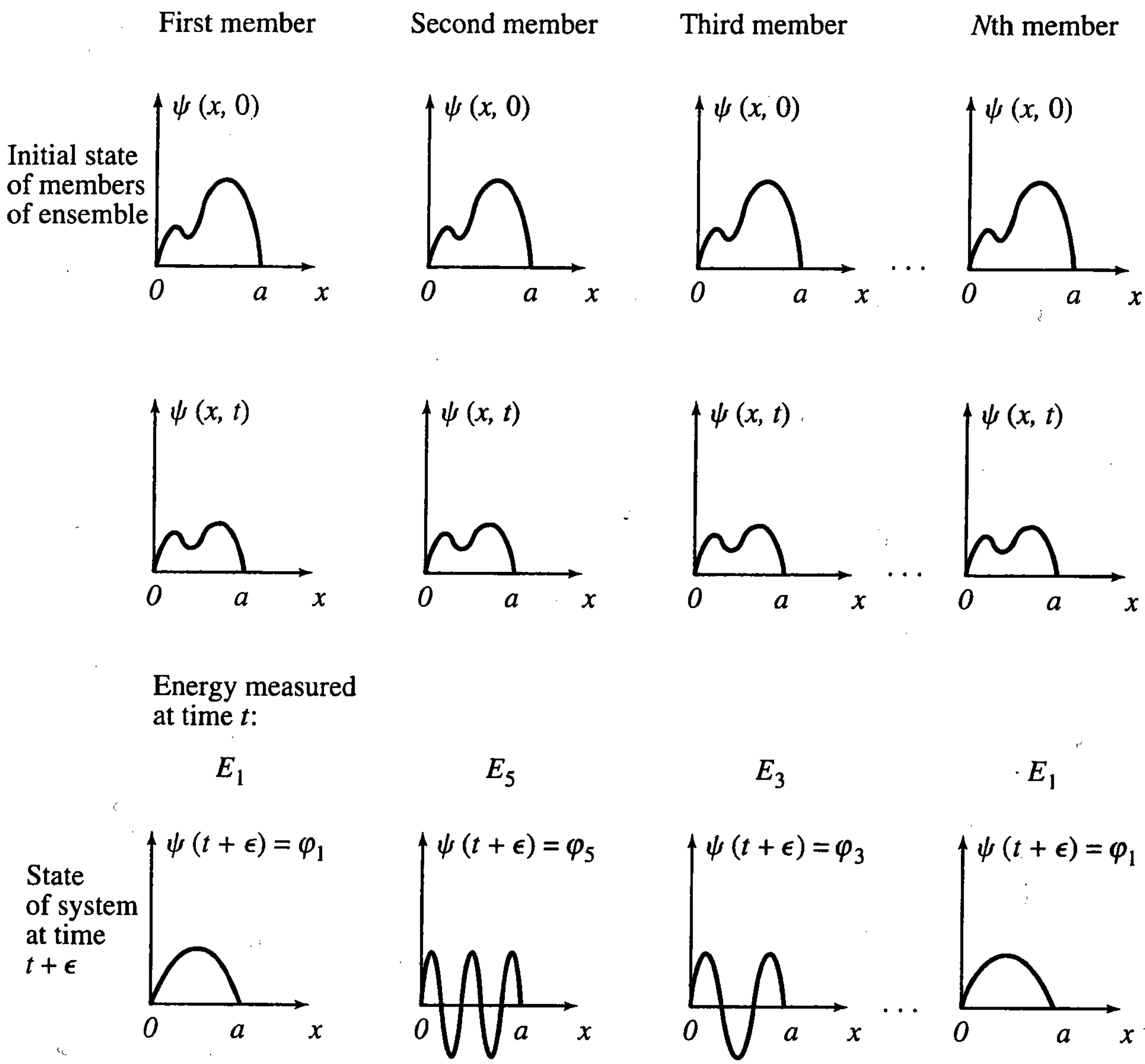


FIGURE 5.1 Measurement of energy of N identical one-dimensional boxes which comprise an "ensemble." All boxes are in the same state at $t = 0$.

the expression

$$\langle E \rangle = \sum_{\text{all } E_n} P(E_n) E_n \tag{5.1}$$

[Recall (3.34)]. This formula holds for all physical observables. For example, the average particle position is given by

Now, the assumption that Ψ is not an energy eigenstate is important to ensure generality.

<u>Question</u>: What happens if we measure the energy of these systems?

<u>Answer</u>: Our QM postulates tell us that each system will give us some energy eigenvalue, and that the system will from then on be in the eigenstate corresponding to that eigenvalue.

But the crucial point is that <u>not</u> all the systems will give the <u>same energy</u> eigenvalue, even they were identical at $t=0$! That's Quantum Mechanics for you. (See 3rd set of plots on 4.2)

So obviously we can't ask about <u>the</u> energy of the state, but we <u>can</u> ask

1) What is the average energy of all the measurements?
2) What is the probability of finding a particular energy eigenvalue?

Well, $\langle E\rangle = \langle \Psi | \hat{H} \Psi \rangle = \underbrace{\sum_{\text{all } E_n} P(E_n)\, E_n}$

We can get the probabilities if we can write the expectation value in this form.

To solve this, expand Ψ in eigenstates of $\hat{H}$. (4.4)
We know we can do this because we already said the particle in a box eigenfunctions form a basis for $\mathcal{H}_1$, subject to the b.c.'s $\Psi=0$ at edge of box.

So we'll use

$$\Psi(x,t) = \sum_{n=1}^{\infty} b_n(t)\, \varphi_n(x)$$

where $\hat{H}\varphi_n = E_n \varphi_n$ + $\varphi_n = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}$ for this case.

In Dirac notation,

$$|\Psi\rangle = \sum_{n=1}^{\infty} |b_n \varphi_n\rangle$$

and

$$\langle E\rangle = \langle \Psi | \hat{H} \Psi\rangle = \left\langle \sum_n b_n\varphi_n \middle| \hat{H} \sum_\ell b_\ell \varphi_\ell \right\rangle$$

$$= \sum_n \sum_\ell b_n^* b_\ell \langle \varphi_n | \hat{H} \varphi_\ell\rangle$$

$$= \sum_n \sum_\ell b_n^* b_\ell E_\ell \underbrace{\langle \varphi_n | \varphi_\ell\rangle}_{\delta_{n\ell}}$$

$$= \sum_{n=1}^{\infty} |b_n|^2 E_n$$

and identifying this with $\langle E\rangle = \sum_n P(E_n) E_n$

we find

$$\boxed{P(E_n) = |b_n|^2}$$

So the prob. of finding a particular ~~energy~~ eigenvalue is just the absolute square of the expansion coefficient for the corresponding energy eigenstate! It all hangs together!

Note: I'm assuming that ψ is properly normalized. If that's true, then all the $P(E_n)$, that is all the $|b_n|^2$, should add up to 1:

$$1 = \langle\psi|\psi\rangle = \langle \sum_n b_n \varphi_n | \sum_\ell b_\ell \varphi_\ell \rangle$$

$$= \sum_n \sum_\ell b_n^* b_\ell \underbrace{\langle \varphi_n | \varphi_\ell \rangle}_{\delta_{n\ell}}$$

$$= \sum_n |b_n|^2$$

And recall that b_n is just the projection (inner prod) of ψ onto φ_n:

$$b_n = \langle \varphi_n | \psi \rangle$$

Other observables: Now suppose we want to ask the same questions (average value, prob. of any given value) about some other observable, like momentum?

Note that we do the same thing we did above, with one modification: substitute the other observable for the energy everywhere, including the eigenstates we use for the expansion of ψ. So let the observable have operator $\hat{F}$ and eigenvalues f_n, with eigenfunctions φ_n (so now φ_n means something different from the energy eigen~~values~~ functions we used it for ~~before~~):

(4.6)

$$\hat{F}\varphi_n = f_n \varphi_n$$

& write $\psi = \sum b_n \varphi_n$ with $b_n = \langle \varphi_n | \psi \rangle$.

Then $\langle f \rangle = \sum |b_n|^2 f_n$ and $|b_n|^2$ is probability of measuring f_n.

Example In general, in a Hilbert space we have basis fns $\{\varphi_n\}$. Let ψ be an arbitrary function in the space

$$\psi = \sum b_n \varphi_n$$

Suppose $\psi = \dfrac{\alpha \varphi_1 + \beta \varphi_2}{\sqrt{\alpha^2+\beta^2}}$ e.g. $\alpha = 3$, $\beta = 4$

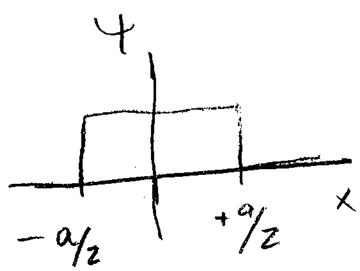
$$b_1 = \langle \varphi_1 | \psi \rangle = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} \; ; \; b_2 = \frac{\beta}{\sqrt{\alpha^2+\beta^2}}$$

The associated probabilities are

$$P_1 = \frac{\alpha^2}{\alpha^2+\beta^2}, \quad P_2 = \frac{\beta^2}{\alpha^2+\beta^2}, \quad P_1 + P_2 = 1 \text{ as it must be}$$

Ex <u>Square wave</u>

Consider a free particle whose wave function is, at $t=0$, a square wave:



$$\psi(x,0) = \begin{cases} 0 & x \le -\frac{a}{2}, \; x \ge +\frac{a}{2} \\ \frac{1}{\sqrt{a}} & |x| < a/2 \end{cases}$$

↑ Gives correct normalization,

Measure the momentum at this instant ($t=0$), what possible values could be found, + with what prob?

<u>sol'n</u> : Expand $\psi(x,0)$ in superposition of $\hat{p}$ eigenstates φ_k

$$\varphi_k = \frac{1}{\sqrt{2\pi}} e^{ikx}$$

These are continuous, so the superposition is an integral

$$\psi(x,0) = \int_{-\infty}^{\infty} b(k)\varphi_k(x)\, dk$$

The $b(k)$ will give the probability density for finding momentum $\hbar k$.

The $b(k)$ are

$$b(k) = \langle \varphi_k | \psi(x,0) \rangle = \int_{-\infty}^{\infty} \varphi_k^*(x)\, \psi(x,0)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x,0)\, e^{-ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi a}} \underbrace{\int_{-a/2}^{a/2} e^{-ikx}\, dx}$$

$$\frac{1}{-ik} e^{-ikx}\Big|_{-a/2}^{+a/2} = \frac{1}{ik}\Big(\underbrace{e^{ika/2} - e^{-ika/2}}_{2i\,\sin ka/2}\Big)$$

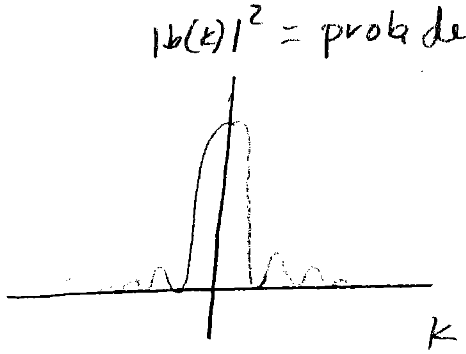
$$= \sqrt{\frac{2}{\pi a}}\ \frac{\sin(ka/2)}{k}$$

So this is the projection of ψ onto momentum eigenstate φ_k. Note that it is an oscillation modulated by $1/k$, so the amplitude gets smaller as k gets bigger. Also, it is finite, not 0, at $x = 0$ (cf. l'Hôpital)

So $$|b(k)|^2 = \frac{2}{\pi a}\ \frac{\sin^2(ka/2)}{k^2}$$

and $|b(k)|^2 dk$ = prob of finding momentum btwn $\hbar k$ & $\hbar(k + dk)$

This looks like



Not only is it not 0 at $k=0$, $|b|^2$ is <u>max</u> at $k=0$.

Then it falls to 0 for $\frac{ka}{2}=\pi$ or $p=\frac{2\pi\hbar}{a}$

So most probable measurement of momentum is $p=0$,

and values w/ $p=\pm n\frac{2\pi\hbar}{a}$ are never found.

Notice that the momentum is most likely to be found in the range w/

$$\Delta p = \hbar\Delta k = \frac{4\pi\hbar}{a}$$

and since it's equally likely to find x anywhere in the interval,

$$\Delta x = a$$

$$\Rightarrow \Delta x \Delta p \simeq 4\pi\hbar \quad \text{or} \quad O(\hbar)$$

Chopped beam:

4.70

Now let's consider a collection of N particles, electrons, say, with this same square wave function (p. 4.7) ψ 

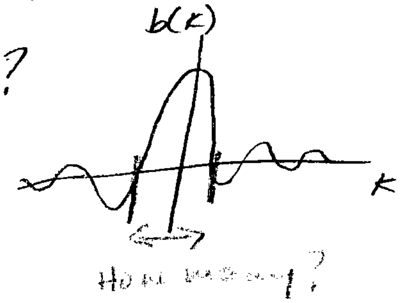
Let there be N of them, and let them not interact with each other. (That means we can treat them as an ensemble, say)

We know that these particles have momenta mostly clustered around $p=0$. We can think of this as a truncated, or chopped, beam of electrons.

We can quantify this clustering by asking:

Q: How many electrons have momenta that fall in that central peak?

(i.e. w/ $-\frac{2\pi\hbar}{a} < p < \frac{+2\pi\hbar}{a}$)



A: To answer, we need the prob. density for the number of particles, which is just N times single particle density $|\psi|^2$

No. particles in $dx = \rho^{(x)} dx = N|\psi|^2 dx$

and to confirm normalization: number in "beam" is

$$N = \int_{-\infty}^{\infty} \rho(x)\,dx = N\int_{-a/2}^{a/2} |\psi|^2 dx = N \text{ ok}$$

Now we need the momentum probability density.

For a single particle it's just $\rightarrow$

prob. of finding momentum btwn $\hbar k$ + $\hbar k + \hbar dk$

= prob of finding k btwn k + $k+dk$

$= P(k)dk = |b(k)|^2 dk$

+ note we're assuming the $b(k)$'s are properly normalized so that $\int b(k)dk = 1$

which is true as long as the original $\psi(x,0)$ is properly normalized.

So for the beam, the momentum prob. density is $\rho(k)\,dk = N|b(k)|^2 dk$

+ it's normalized, like $\rho(x)$, so that integrating over all k gives the total no. of particles in the beam:

$$N = \int_{-\infty}^{\infty} \rho(k)dk = N\int_{-\infty}^{\infty} |b(k)|^2 dk = N$$

Finally, to answer the question, the no. ΔN of particles w/ mom. in the peak $\left(-\frac{2\pi\hbar}{a} < p < \frac{2\pi\hbar}{a}\right.$ or equivalently $\left.-\frac{2\pi}{a} < p < \frac{2\pi}{a}\right)$ is

$$\Delta N = N\int_{-2\pi/a}^{2\pi/a} |b(k)|^2 dk = N\int_{-2\pi/a}^{2\pi/a} \frac{2}{\pi a}\,\frac{\sin^2(ka/2)}{k^2}\,dk$$

+ w/ $\eta \equiv ka/2$, $k^2 = \left(\frac{2\eta}{a}\right)^2$, $dk = \frac{2}{a}d\eta$ $\rightarrow$

$$\Delta N = \frac{N}{\pi}\int_{-\pi}^{+\pi} \frac{\sin^2\eta}{\eta^2}\,d\eta = 0.903N$$

$\Rightarrow$ most of the particles have momenta in this interval. And note that the answer is independent of a.

Superposition + uncertainty: Let's make sure we're still consistent w/ the uncertainty principle

① Suppose we take the square wave, + at $t=0$ measure p. (can't get $p = \pm n2\pi\hbar/a$)

Suppose we measure $p = \pi\hbar/a$

Immediately after, particle is in eigenstate for this p: $\psi = \frac{1}{\sqrt{2\pi}}\, e^{ipx/\hbar} = \frac{1}{\sqrt{2\pi}}\, e^{i\pi x/a}$

Now measure E. This ψ is eigenstate of H as well, so get

$$E = \frac{(\pi\hbar/a)^2}{2m}$$

Now suppose we measure x. Prob density is

$$P = |\psi|^2 = \frac{1}{2\pi}$$

$\Rightarrow$ all values of x equally likely. $\Delta x = \infty$

But p was definite so $\Delta p = 0$, consistent w/ uncertainty principle.

(2) Now take this momentum eigenstate and measure x. Suppose we get $x = x'$ (4.13)

Then what is the state of the particle? According to our postulates, it must be the corresponding eigenstate of x

$$\psi = \delta(x - x')$$

(No longer the momentum eigenstate.)

So now what happens if we measure P? well, we can answer that by using the superposition idea: write ψ as sum of eigenstates of P.

So $$\psi = \delta(x - x') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} b(k)\, e^{ikx}\, dk$$

& the $|b(k)|^2$ give the prob. for finding the corresponding P's (because $p = \hbar k$).

$$b(k) = \langle \varphi_k | \psi \rangle = \frac{1}{\sqrt{2\pi}} \int \delta(x - x') e^{-ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-ikx'}$$

and prob is $P(k) = |b(k)|^2 = \frac{1}{2\pi}$

all values of momentum equally likely, so $\Delta p = \infty$ and $\Delta x = 0$, still ok for unc. princ.

So superposition means any given state can be thought of as being in a combination of other states:

eigenstate of x is sum of p eigenstates

e.s. of p —"— x e.s.'s

Commutator Relations

I've made a point a few times of the fact that operators don't always give the same thing when applied in different orders, i.e., they don't always commute. Think of $\hat{x}$ and $\hat{p}$ — because $\hat{p}$ involves a derivative, it matters whether it acts before or after $\hat{x}$.

We formalize this notion by defining the commutator, which is the difference between a pair of operators acting in one order vs the other:

$$\boxed{[\hat{A},\hat{B}] \equiv \hat{A}\hat{B}-\hat{B}\hat{A}} \quad \text{Commutator}$$

Note that $[\hat{B},\hat{A}] = -[\hat{A},\hat{B}]$

If the order of operators $\hat{A}$ & $\hat{B}$ doesn't matter, their commutator is 0 & $\hat{A}$ & $\hat{B}$ are said to commute

$$\hat{A}\hat{B}=\hat{B}\hat{A} \Leftrightarrow [\hat{A},\hat{B}] = 0 \Leftrightarrow \hat{A} \text{ \& } \hat{B} \text{ commute}$$

Also, Liboff likes to say the corresponding →

observables A + B are <u>compatible</u>.

<u>Some properties and examples:</u>

$$a = \text{const} \Rightarrow [\hat{A}, a] = 0$$

$$[\hat{A}, a\hat{B}] = [a\hat{A}, \hat{B}] = a[\hat{A}, \hat{B}]$$

Any operator commutes with itself: $[\hat{A}, \hat{A}] = \hat{A}\hat{A} - \hat{A}\hat{A} = 0$

Any operator commutes w/ its own square

$$[\hat{A}, \hat{A}^2] = \hat{A}\hat{A}\hat{A} - \hat{A}\hat{A}\hat{A} = 0$$

This implies momentum commutes w/ kinetic energy $\frac{\hat{p}^2}{2m}$, i.e. $\hat{p}$ commutes w/ $\hat{H}$ for a free particle.

and indeed $\hat{A}$ commutes with any power of $\hat{A}$, from which it follows that $\hat{A}$ commutes with any function of itself.

$$[f(\hat{A}), \hat{A}] = 0$$

(think power series expansion of f.)

Comment: A commutator being zero means that if it (the commutator) acts on any function - call it $g(x)$ - you get zero;

$$[f(\hat{A}), \hat{A}]\, g(x) = 0, \quad \text{any } g(x)$$

Note that from this we have

$$[e^{\hat{p}}, \hat{p}] = \left[\sum_{n=0}^{\infty} \frac{\hat{p}^n}{n!}, \hat{p}\right] = \sum_{n=0}^{\infty} \frac{1}{n!}[\hat{p}^n, \hat{p}]$$

$$= [1, \hat{p}] + [\hat{p}, \hat{p}]$$

Ex Speaking of $\hat{x}$ + $\hat{p}$, let's find their commutator:

$$[\hat{x},\hat{p}]g(x) = i\hbar\left(-x\frac{\partial}{\partial x} + \frac{\partial}{\partial x}\,x\right)g(x)$$

note this acts on everything that follows, not just x

$$= i\hbar\left[\cancel{-x\frac{\partial g}{\partial x}} + g(x) + \cancel{x\frac{\partial g}{\partial x}}\right]$$

$$= i\hbar\, g(x)$$

which means

$$\boxed{[\hat{x},\hat{p}] = i\hbar}$$

One of the all-time important commutators in physics.

There are plenty of other things you can show about operators. For example

$$[\hat{A}+\hat{B},\hat{C}] = [\hat{A},\hat{C}] + [\hat{B},\hat{C}]$$

$$[\hat{A}\hat{B},\hat{C}] = \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B}$$

+ from the latter follows

$$[\hat{x},\hat{p}^2] = [\hat{x},\hat{p}]\hat{p} + \hat{p}[\hat{x},\hat{p}]$$

$$= 2i\hbar\hat{p}$$

(+ this implies that $\hat{x}$ does not commute w/ kinetic energy + hence $\hat{x}$ does not commut w/ $\hat{H}_{free}$.)

So $[\hat{x}, \hat{p}^2] g(x) = 2\hbar^2 \frac{\partial g}{\partial x}$

Similarly $[\hat{x}^2, \hat{p}] = \hat{x}[\hat{x}, \hat{p}] + [\hat{x}, \hat{p}]\hat{x} = 2i\hbar x$

Theorem If $\hat{A}$ & $\hat{B}$ commute, they have a set of common non-trivial eigenfunctions. So if $[\hat{A}, \hat{B}] = 0$, we can find a set of functions that are eigenfunctions of both!

[We've already seen an example: $\hat{p}$ commutes with the free particle Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m}$, and they both have eigenfns $e^{ikx} = e^{ipx/\hbar}$]

Proof Let $\hat{A}$ have eigenfns φ_a w/ eigenvalues a:

$$\hat{A}\varphi_a = a\varphi_a$$

Then $\hat{B}\hat{A}\varphi_a = a\hat{B}\varphi_a$

but $\hat{B}\hat{A} = \hat{A}\hat{B}$ so we can write

$$\hat{A}(\hat{B}\varphi_a) = a(\hat{B}\varphi_a)$$

which is some function

which says $(\hat{B}\varphi_a)$ is an eigenfunction of $\hat{A}$ with eigenvalue a. Now suppose φ_a is

the fine point → the only linearly independent eigenfn of $\hat{A}$ w/ eigenvalue a. Then the function $\hat{B}\varphi_a$ must be a constant times φ_a, i.e.

$$\hat{B}\varphi_a = \mu\varphi_a$$

const.

but that equation says ψ_a is an eigenfn of $\hat{B}$ also! 4.18

end of prf

Returning to our $\hat{p}$, $\hat{H}$ example,

$$\hat{p}\, e^{ikx} = \hbar k\, e^{ikx}$$

$$\hat{H}\, e^{ikx} = \frac{\hbar^2 k^2}{2m} e^{ikx}$$

Degeneracy: Now let's deal with the fine print in the above proof, i.e the part about ψ_a being the only linearly independent eigenfn of $\hat{A}$. What if it's not? Then we have degeneracy.

Aside: linear independence
So remember what linear independence means. The set $\{\psi_n\}$ of N functions is linearly independent only if

$$\sum_{n=1}^{N} \lambda_n \psi_n = 0$$

can be solved for $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$

Ex: e^x + $\sin x$ are linearly independent:

$$\lambda_1 e^x + \lambda_2 \sin x = 0 \quad \text{only for } \lambda_1 = \lambda_2 = 0$$

By contrast, e^x + $3e^x$ are not linearly indep because

$$\lambda_1 e^x + \lambda_2 3e^x = 0 \quad \text{for } \lambda_1 = -3\lambda_2 \neq 0$$

Notice that if ψ_a is the only linearly indep. eigenfn of $\hat{A}$ w/ e.v. = a, then any multiple of ψ_a is also an eigenfn w/ e.v.=a, but not linearly indep.

On to degeneracy. Let's say ψ_a is not the only linearly indep eigenfn of $\hat{A}$ w/ eigenvalue a. For simplicity, let's say there are two such fns; call them φ_1 + ψ_2:

$$\hat{A}\varphi_1 = a\varphi_1$$

$$\hat{A}\psi_2 = a\psi_2$$

and the eigenvalue a is said to be degenerate, and in this case it happens to be doubly degenerate. (Guess what triply degenerate means?)

So what's the most general eigenfunction of $\hat{A}$ w/ eigenvalue a? It's a combination of φ_1 + ψ_2

$$\psi_a = \alpha\varphi_1 + \beta\psi_2$$

Now getting back to the theorem and its proof, suppose $\hat{A}$ + $\hat{B}$ commute and we have

$$\hat{B}\hat{A}\varphi_1 = a(\hat{B}\varphi_1) = \hat{A}\,(\hat{B}\psi_1)$$

so $\hat{B}\psi_1$ is an eigenstate of $\hat{A}$ w/ eigenvalue a as before. But now that $\hat{A}$ is doubly degenerate, the most we can say is

$$\hat{B}\varphi_1 = \mu(\alpha\varphi_1 + \beta\psi_2)$$

and it no longer follows that ψ_1 is an eigenfn of $\hat{B}$

To summarize, If $[\hat{A}, \hat{B}] = 0$, and a is a degenerate eigenvalue of $\hat{A}$, the corresponding eigenfns of $\hat{A}$ need not be eigenfns of $\hat{B}$. (4.20)

Ex A physical example: the same one as above! Free particle in one dimension.

$$\hat{H} = \frac{\hat{P}^2}{2m}$$

The eigenvalue $E_K = \frac{\hbar^2 k^2}{2m}$

is doubly degenerate. It has eigenfns

$$\cos Kx, \ \sin kx, \ e^{ikx}$$

all w/ the same eigenvalue E_K. [Aside: I listed 3 eigenfns. So why not triply degenerate? Because the 3 eigenfns are not linearly independent: $e^{ikx} = \cos kx + i \sin kx$]

Now, we already saw that $[\hat{p}, \hat{H}] = 0$ for a free particle. But the eigenfns $\{\cos kx, \sin kx\}$ are doubly degenerate, so by the fine print in the commutator thm, they need not be eigenfns of $\hat{p}$. They're not, in fact:

$$\hat{p}(\cos kx) = -i\hbar \frac{\partial}{\partial x} \cos kx = i\hbar K \sin kx \neq \text{const} * \cos kx$$

So where does that leave us? Well, we can write down a linearly indep set of eigenfns of both $\hat{p}$ + $\hat{H}$

→

e^{ikx}, e^{-ikx} eigenfns of $\hat{p}, \hat{H}$

Degenerate in $\hat{H}$, not in $\hat{p}$.

We can generalize this to create an addendum to the commutator thm:

If $[\hat{A}, \hat{B}] = 0$ and $\hat{A}$ has an n-fold degenerate set of eigenstates, we can form n linear combinations of these which are linearly independent eigenstates of both $\hat{A}$ + $\hat{B}$.

<u>Commutator Rel'ns and the Uncertainty Principle</u>

There is a deep connection between the compatability of observables (viz., whether they commute w/ each other) and how well they can be measured simultaneously.

<u>EX</u>: Free particle in 1-dim. Functions $\varphi = e^{ikx}$ are eigenfns of both $\hat{p}$ + $\hat{H}$, which have $[\hat{p}, \hat{H}] = 0$.
Now take a particle in this state, and measure its momentum. You get $p = \hbar k$ with $\Delta p = 0$ and it remains in the same state. Now measure E.
You get $E = \frac{\hbar^2 k^2}{2m}$ with $\Delta E = 0$.

The point is $[\hat{p}, \hat{H}] = 0$ and we can have $\Delta p \Delta E = 0$, i.e. we can measure both simultaneously

Ex Now consider position and momentum, $[X,P]\neq 0$. As we've discussed before, we can't measure them both with zero uncertainty.

$$[\hat{X},\hat{P}] = i\hbar \quad \text{and} \quad \Delta p \Delta x > \frac{\hbar}{2}$$

This generalizes. For a state ψ if we measure $\hat{A}$ w/ uncertainty ΔA, then measurement of $\hat{B}$ has unc. ΔB such that

$$\boxed{\Delta A\, \Delta B \geq \frac{1}{2}\left|\langle [A,B] \rangle\right|}$$

r.m.s. deviation

expectation value in state ψ

A way to understand this: Measuring A leaves the system in an eigenstate of $\hat{A}$. If $\hat{A}$ + $\hat{B}$ do not commute, this state is not an eigenstate of $\hat{B}$, but it can be written as a linear comb. of $\hat{B}$ eigenstates. So there is some spread of values allowed for a measurement of B, and some corresponding uncertainty

Complete sets of commuting observables

We can combine some of the above ideas into the notion that a quantum state can be uniquely specified by some set of observables which commute w/ each other.

These are called <u>complete sets of commuting observables</u>.

For example, consider (again) a free particle in 1-Dim. If we specify the energy, $E = \frac{\hbar^2 k^2}{2m}$, that narrows down the possibilities but doesn't uniquely specify the state: it could be e^{+ikx} or e^{-ikx}. But if we then also specify the momentum, that uniquely specifies the state. And there's no other observable that can give use more info about the state.

Another example is the hydrogen atom. If we specify the principal quantum number n, that tells us the energy and the radial part of the wave function, but to uniquely specify the total wave function we also need to know the total orbital angular momentum (quantum number l) and z component of angular momentum (m_l). (I'm ignoring spin.)

More generically, a system is uniquely specified by some complete set of commuting observables $\hat{A}, \hat{B}, \hat{C} \ldots$ (the numbers of these can vary) which can be measured simultaneously. The corresponding values $a, b, c \ldots$ are sometimes called "good quantum numbers"