[Angular Momentum] Liboff Chapter 9

In this chapter we're going to discuss angular momentum, which we introduced in the context of the hydrogen atom in P237. In fact everything we said there about orbital angular momentum applies to any problem with spherical symmetry!

Recall
$$\vec{L} = \vec{r} \times \vec{p}$$
$$= -i\hbar(\vec{r} \times \vec{\nabla})$$

and we found (in spherical coord's)
$$L^2 = \vec{L} \cdot \vec{L} = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]$$

and when we write ∇^2 in spherical coords,
$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{\hbar^2}\frac{L^2}{r^2}$$

$\Rightarrow$ all of the angular dependence in the Schr. eq'n for a spherically symmetric potential is determined by the L^2 operator.
The angular part of the sol'ns were eigenfns of L^2 (and, as it turned out, L_z).
And L^2 and L_z eigenvalues were quantized:
$$L^2 = \hbar^2 l(l+1) \qquad l = 0, 1, 2, \ldots$$
$$L_z = \hbar m_l \qquad m_l = -l, -l+1, \ldots l-1, l$$

Now we're going to show how we can obtain all of the same results, and many more, without using the differential equation machinery we used in P237 to solve the Schrödinger eq'n. Instead, we'll make further use of the kind of operator formalism we found so powerful in solving the simple harmonic oscillator.

Basic properties of angular momentum

We'll start with orbital angular momentum, then generalize to include spin.

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\boxed{\vec{L} = -i\hbar\, \vec{r} \times \vec{\nabla}}$$

so cartesian components are

$$\hat{L}_x = -i\hbar \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right)$$

$$\hat{L}_y = -i\hbar \left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right)$$

$$\hat{L}_z = -i\hbar \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)$$

Commutation rel'ns: Since the components of $\vec{L}$ are differential operators we don't necessarily expect them to commute with each other. As we'll see, they don't, which means

You can't simultaneously specify the components of angular momentum in a single state

Mathematically, it means L_x, L_y, + L_z don't have common eigenfunctions.

Let's calculate the commutators explicitly: (7.3)

$$[\hat{L}_x, \hat{L}_y] = \hat{L}_x \hat{L}_y - \hat{L}_y \hat{L}_x$$

cancel

cancel

$$= (yp_z - zp_y)(zp_x - xp_z) - (zp_x - xp_z)(yp_z - zp_y)$$

$$= yp_z zp_x + zp_y xp_z - zp_x yp_z - xp_z zp_y$$

$$= yp_x(\underbrace{p_z z - zp_z}_{-i\hbar}) + xp_y(\underbrace{zp_z - p_z z}_{i\hbar})$$

$$= i\hbar \hat{L}_z$$

Similarly for the other two permutations, to give

$$\boxed{\begin{aligned} [\hat{L}_y, \hat{L}_z] &= i\hbar \hat{L}_x \\ [\hat{L}_z, \hat{L}_x] &= i\hbar \hat{L}_y \\ [\hat{L}_x, \hat{L}_y] &= i\hbar \hat{L}_z \end{aligned}}$$

or in vector notation

$$i\hbar \vec{L} = \vec{L} \times \vec{L}$$

So since no two components of $\vec{L}$ commute, you can specify the value of, at most, <u>one</u> of the components of $\vec{L}$, say L_z. But it turns out that you can also specify the value of $L^2 = \vec{L}\cdot\vec{L}$. For this to be true, we must have $[\hat{L}_z, \hat{L}^2] = 0$

$$[\hat{L}_z, \hat{L}^2] = [L_z, L_x^2 + L_y^2 + L_z^2]$$

& with $[L_z, L_z^2] = 0$

$$= [L_z, L_x^2] + [L_z, L_y^2]$$

& with $[a, b^2] = [a,b]b + b[a,b]$

$$= \underbrace{[L_z, L_x]}_{i\hbar L_y} L_x + L_x \underbrace{[L_z, L_x]}_{i\hbar L_y} + \underbrace{[L_z, L_y]}_{-i\hbar L_x} L_y + L_y \underbrace{[L_z, L_y]}_{-i\hbar L_x}$$

$$= i\hbar [L_y L_x + L_x L_y - L_x L_y - L_y L_x]$$

$$= 0$$

Similarly, each component of $\vec{L}$ commutes w/ L^2:

$$[L_x, L^2] = [L_y, L^2] = [L_z, L^2] = 0$$

$$\text{or } [\vec{L}, L^2] = 0$$

So L^2 and each of its components separately have common eigenfns (e.g. L^2 and L_z) but no more than that.

Suppose we have common eigenfns of L^2 & L_z: call them ψ_{lm}. Then

$$L^2 \psi_{lm} = \hbar^2 l(l+1)\psi_{lm} \qquad l = 0, 1, 2, \ldots$$

$$L_z \psi_{lm} = \hbar m \psi_{lm} \qquad m = -l, -l+1, \ldots +l$$

(we take the identity of the eigenvalues as given for the moment; we'll show below that l and m are indeed what we claim.)

So if we have a system and we measure its total angular momentum L and the z component L_z, then the possible values are

$$L = \hbar\sqrt{l(l+1)} \quad ; \quad L_z = \hbar m$$

And notice that the eigenvalues of $\underline{L}^2$ are degenerate: for each value of l there are $2l+1$ possible values of m.

EX Suppose we measure the angular momentum of a (isolated) wheel, and suppose we find $L^2 = 30\hbar^2$. This means $l = 5$. Note that once we've made this measurement, the wheel is left in an eigenstate of L^2 with this eigenvalue. (But it may or may not be in an eigenstate of L_z.)

Now measure L_z. Possible values are

$$L_z = -5\hbar, -4\hbar, \ldots 0, \hbar, \ldots 4\hbar, 5\hbar$$

Suppose we find $L_z = 3\hbar \Rightarrow m = 3$. Then the wheel is left in an eigenstate of L_z as well. After both measurements, the wheel is in the state $\psi_{5,3}$

Uncertainty Relations: We can't get more angular momentum info. out of a system than L^2 and one of its components, say L_z. If we measure L^2 and L_z, the system is left in an eigenstate of the two. But then if we measure L_x, the system is no longer in an eigenstate of L_z and we lose all info about the value of L_z. Similarly for L_y. This is all contained in the uncertainty rel'n for ang. mom.

Recall $\Delta A \ \Delta B \geq \frac{1}{2} |\langle [A, B] \rangle|$

which in this case gives, for example

$$\Delta L_y \Delta L_z \geq \frac{\hbar}{2} |\langle L_x \rangle|$$

and if we've just measured $L_x = 5\hbar$, say, the rel'n becomes $\Delta L_y \Delta L_z \geq \frac{5\hbar^2}{2}$

Orbital vs Spin ang. mom.: You will recall from P237 that in addition to orbital ang. mom. there's another kind of ang. momentum which is intrinsic to particles and it's called spin. Spin is also quantized, though in half-integer multiples of $\hbar$.

So we'll now switch to talking about angular momentum generically.

4/15/03

Last time

Ang. momentum eigenvalues

$[J_x, J_y] = i\hbar J_z$ etc $\quad [J_i, J^2] = 0$

Define
$J_+ = J_x + iJ_y$

$J_- = J_x - iJ_y$

$\Rightarrow \quad [J_z, J_\pm] = \pm\hbar J_\pm$

$[J^2, J_\pm] = 0$

$[J_+, J_-] = 2\hbar J_z$

$J^2 = J_+ J_- + J_z^2 - \hbar J_z$

$= J_- J_+ + J_z^2 + \hbar J_z$

$\Rightarrow J_\pm \psi_m = \text{const}\ \psi_{m\pm1}$

J^2, J_z common e.s's

Today
Ang mom. eigenfunctions

Angular Momentum Operator and its Eigenvalues

We'll now use the operator formalism to find the eigenvalues of ang. mom.

Notation: $\vec{J}$ = generic ang. mom.

$\vec{L}$ = orbital ang. mom.

$\vec{S}$ = spin ang. mom.

Defining relations:

$$[J_x, J_y] = i\hbar J_z$$

$$[J_y, J_z] = i\hbar J_x$$

$$[J_z, J_x] = i\hbar J_y$$

$$J^2 = \vec{J}\cdot\vec{J} = J_x^2 + J_y^2 + J_z^2$$

and from the defining rel'ns it follows

$$[J_x, J^2] = [J_y, J^2] = [J_z, J^2] = 0$$

$$\Delta J_x \, \Delta J_y \geq \frac{\hbar}{2} |\langle J_z \rangle|$$

Raising + lowering operators: These are similar to $\hat{a}, \hat{a}^+$

Let $\hat{J}_+ \equiv \hat{J}_x + i\hat{J}_y$

$\hat{J}_- = \hat{J}_x - i\hat{J}_y = J_+^\dagger$

Now a bunch of commutation, etc, relations

$$[J_z, J_+] = [J_z, J_x] + i[J_z, J_y]$$

$$= i\hbar J_y + i(-i\hbar J_x)$$

$$= \hbar (J_x + iJ_y)$$

$$= \hbar J_+$$

similarly, $[J_z, J_-] = -\hbar J_-$

$$\Rightarrow \boxed{[J_z, J_\pm] = \pm \hbar J_\pm}$$

$$[J^2, J_+] = [J^2, J_x] + i[J^2, J_y] = 0$$

similarly, $[J^2, J_-] = 0$

$$\Rightarrow \boxed{[J^2, J_\pm] = 0}$$

$$[J_+, J_-] = [J_x + iJ_y, J_x - iJ_y]$$

$$= 2i\underbrace{[J_y, J_x]}_{-i\hbar J_z}$$

$$= 2\hbar J_z$$

$$\Rightarrow \boxed{[J_+, J_-] = 2\hbar J_z}$$

And writing J^2 in terms of J_+ + J_-:

$$J_+ J_- = (J_x + iJ_y)(J_x - iJ_y)$$

$$= J_x^2 + J_y^2 + i\underbrace{[J_y J_x - J_x J_y]}_{-i\hbar J_z}$$

$$= J_x^2 + J_y^2 + \hbar J_z$$

$$\Rightarrow J^2 = J_+ J_- + J_z^2 - \hbar J_z$$

+ similarly $J^2 = J_- J_+ + J_z^2 + \hbar J_z$

Now we'll use these relations to determine what the eigenvalues of J^2 and J_z are. Or rather to confirm, because we already know the answer. The method uses raising and lowering operators in much the same way as for the simple harmonic oscillator. It goes like this:

Let φ_m be an eigenstate of J_z with eigenvalue $\hbar m$:

$$J_z \varphi_m = \hbar m \varphi_m$$

We know that m will turn out to be an integer or half-odd integer. We want to show this.

What does J_+ do to φ_m? Take $(J_+\varphi_m)$ + apply J_z:

$$J_z (J_+ \varphi_m) = (J_+ J_z + \hbar J_+)\varphi_m \qquad J_z J_+ = J_+ J_z + \hbar J_+$$

$$= \hbar(m+1)(J_+ \varphi_m)$$

$\longrightarrow$

So $(J_+\varphi_m)$ is an eigenfunction of J_z with eigenvalue $\hbar(m+1)$. $\Rightarrow$ J_+ is indeed a raising operator:

$$J_+\varphi_m = \overset{\text{const}}{\wedge}\varphi_{m+1}$$

$$J_+\varphi_{m+1} = \overset{\text{const}}{\wedge}\varphi_{m+2} \quad \text{etc}$$

Similarly,

$$J_-\varphi_m = \text{constant} * \varphi_{m-1}$$

$$J_-\varphi_{m-1} = \text{const} * \varphi_{m-2} \quad \text{etc}$$

So using $J_\pm$, we can generate a sequence of J_z eigenfns w/ values of m differing by one.

Now recall J^2 commutes w/ J_z, so they have common eigenfunctions. Let φ_m be such a common e.f. w/ J^2 eigenvalue $\hbar^2 K^2$; we'll figure out what K^2 is below. So

$$J^2\varphi_m = \hbar^2 K^2 \varphi_m$$

Now operate w/ J_+:

$$J_+ J^2 \varphi_m = J_+ \hbar^2 K^2 \varphi_m$$

but since $[J^2, J_+] = 0$, $J_+J^2\varphi_m = J^2(J_+\varphi_m)$

or,
$$J^2(J_+\varphi_m) = \hbar^2 K^2 (J_+\varphi_m)$$

$\longrightarrow$

$$\Rightarrow J^2\varphi_{m+1} = \hbar^2 K^2 \varphi_{m+1}$$

$\Rightarrow$ φ_{m+1} is also an eigenfunction of J^2 with the same J^2 eigenvalue as φ_m. That means the <u>entire</u> sequence obtainable from J_+ + J_- are eigenfns of J^2 w/ the same eigenvalue $\hbar^2 \underline{K}^2$.

That lets us figure out that m must have a maximum value, which follows if we take $\langle J^2 \rangle$ in the state φ_m:

$$\langle J^2 \rangle = \hbar^2 \underline{K}^2 = \langle J_x^2 \rangle + \langle J_y^2 \rangle + \underbrace{\langle J_z^2 \rangle}_{\hbar^2 m^2}$$

$$\Rightarrow \hbar^2 \underline{K}^2 = \langle J_x^2 \rangle + \langle J_y^2 \rangle + \hbar^2 m^2$$

Now J_x + J_y are Hermitian, so $\langle J_x^2 \rangle \geq 0$ + $\langle J_y^2 \rangle \geq 0$

$$\Rightarrow \quad \hbar^2 \underline{K}^2 \geq \hbar^2 m^2$$

or $\quad |\underline{K}| \geq |m|$

So given $\underline{K}$, the only possible values in the sequence of $\{\varphi_m\}$ must fall between $+\underline{K}$ + $-\underline{K}$.

So there is some max m, call it m_{max}, for which

$$J_+ \varphi_{m_{max}} = 0$$

Similarly $\exists$ m_{min} with

$$J_- \varphi_{m_{min}} = 0$$

Now recall $J^2 = J_- J_+ + J_z^2 + \hbar J_z$ (cf. p. 7.9) 7.12

Apply to $\varphi_{m_{max}}$:

$$J^2 \varphi_{m_{max}} = \hbar^2 \mathcal{K}^2 \varphi_{m_{max}} = J_- \cancel{J_+ \varphi_{m_{max}}} + J_z^2 \varphi_{m_{max}} + \hbar J_z \varphi_{m_{max}}$$

$$\text{or} \quad \hbar^2 \mathcal{K}^2 \varphi_{m_{max}} = \hbar^2 m_{max}(m_{max}+1)\varphi_{m_{max}} \qquad (*)$$

And using $J^2 = J_+ J_- + J_z^2 - \hbar J_z$ on $\varphi_{m_{min}}$,

$$\hbar^2 \mathcal{K}^2 \varphi_{m_{min}} = \hbar^2 m_{min}(m_{min}-1)\varphi_{m_{min}} \qquad (**)$$

And (*) and (**) imply

$$m_{max}(m_{max}+1) = m_{min}(m_{min}-1)$$

$$\Rightarrow \quad m_{max} = -m_{min}$$

So the possible values of m are a symmetric sequence about 0.

Let $\qquad j \equiv m_{max}$

Then since m goes fr/ $-j$ to $+j$ in unit steps, we must have

$j =$ integer $\Rightarrow$ all m's are integers

or $\quad j = \frac{1}{2}$-odd integer $\Rightarrow$ m's are $\frac{1}{2}$-odd integers and $J^2 = \hbar^2 \mathbb{I}^2 = \hbar^2 m_{max}(m_{max}+1) = \hbar^2 j(j+1)$ (7.13)

So
$$\boxed{\begin{array}{ll} J^2 = \hbar^2 j(j+1) & j = \text{integer, } \tfrac{1}{2} \text{ odd integer} \\ J_z = \hbar m_j & m_j = -j, \dots +j \end{array}}$$

<u>Orbital Angular Momentum Eigenfunctions</u>

The common eigenfns ψ_{lm} of L^2 & L_z are called "spherical harmonics" and as the name suggests are most easily handled in spherical coordinates. Also we'll switch to the standard notation:

$$\psi_{lm} \rightarrow Y_l^m(\theta, \varphi)$$

So the eigenvalue eq'ns become

$$L^2 Y_l^m = \hbar^2 l(l+1) Y_l^m$$
$$L_z Y_l^m = \hbar m Y_l^m$$

We will solve the L_z eq'n to get the φ (azimuthal angle) dependence, as we did in P237. The next thing we did in P237 was to find a series sol'n for the θ dependence (and recall that involved both l and m).

Here we will take an alternate route. In the spirit of our algebraic sol'n for the simple harmonic operator,

$\longrightarrow$ next page

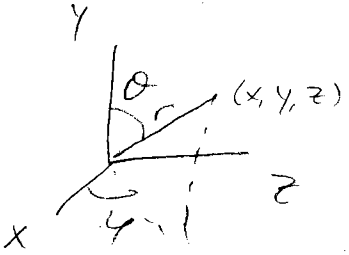
we will solve

$$L_+ Y_\ell^\ell = 0 \qquad \text{for } Y_\ell^\ell$$

and generate the remaining values of m by using the lowering operator:

$$Y_\ell^{\ell-1} = L_- Y_\ell^\ell \quad \text{etc}$$

So in spherical coords



$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$

recall

$$L_x = i\hbar\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)$$
$$L_y = i\hbar\left(-\cos\varphi\frac{\partial}{\partial\theta} + \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)$$
$$L_z = -i\hbar\frac{\partial}{\partial\varphi}$$

and

$$L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]$$

Then

$$L_+ = L_x + iL_y$$
$$= i\hbar\left[(\sin\varphi - i\cos\varphi)\frac{\partial}{\partial\theta} + (\cos\varphi + i\sin\varphi)\cot\theta\frac{\partial}{\partial\varphi}\right]$$

$$L_+ = \hbar e^{i\varphi}\left[\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right]$$

and $L_- = \hbar e^{-i\varphi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)$

Aside Note that the angular momentum op's (and their eigenfns) depend only on θ and φ and not on r. This makes sense when we think back to our discussion of symmetries & conservation laws. Recall that angular momentum is related to rotations (p. 5.25-5.28 in notes; p. 174-175 in book)

In particular we said if we rotate through a small angle α about the z axis, the transformed wave function is

$$\psi(x', y', z) = \psi(x,y,z) + \frac{i}{\hbar}\alpha L_z \psi(x,y,z)$$

and if α is not small it's

$$\psi(x', y', z) = e^{i\alpha L_z/\hbar}\psi(x,y,z)$$

$\Rightarrow L_z$ generates rotations about the z axis; similarly for L_x + L_y

Rotations affect angles, but not r.

Normalization of the $Y_\ell^m(\theta, \varphi)$

The $Y_\ell^m(\theta,\varphi)$ are normalized s.t. the integral of $|Y_\ell^m|^2$ over all solid angle (4π steradians) is $= 1$. Also they're orthogonal; so

(7.16)

$$\int |Y_\ell^m|^2 d\Omega = \int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\varphi \, |Y_\ell^m(\theta,\varphi)|^2 = 1$$

and also $\int d\Omega \, Y_\ell^{m*} Y_{\ell'}^{m'} = \delta_{\ell\ell'}\delta_{mm'}$

Sol'n for φ dependence:

L_z eq'n is $\quad L_z Y_\ell^m(\theta,\varphi) = \hbar m Y_\ell^m(\theta,\varphi)$

or $\quad \frac{\partial}{\partial\varphi} Y_\ell^m(\theta,\varphi) = im Y_\ell^m(\theta,\varphi)$

Factorize the θ + φ dependence:

$$Y_\ell^m(\theta,\varphi) = \Theta_\ell^m(\theta)\,\Phi_m(\varphi)$$

and eq'n becomes

$$\frac{\partial}{\partial\varphi}\Phi_m(\varphi) = im\,\Phi_m(\varphi)$$

$$\Rightarrow \quad \Phi_m(\varphi) = \text{const}\, e^{im\varphi}$$

and if we require that $\int_0^{2\pi} d\varphi\, |\Phi|^2 = 1$ (in which case we must also have $\int_{-1}^{1} d\cos\theta\, |\Theta|^2 = 1$) then

$$\int_0^{2\pi} d\varphi\, |\Phi|^2 = 2\pi\,(\text{const})^2 \Rightarrow \text{const} = \frac{1}{\sqrt{2\pi}}$$

so $$\boxed{\Phi_m(\varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi}}$$

and if we also require that the wave fn be single-valued, so

$$\Phi(\varphi + 2\pi) = \Phi(\varphi)$$

$$\Rightarrow \quad e^{im2\pi} = 1 \Rightarrow m = \text{integer}$$ f... orbital ...

We already know we must have m going from $-l$ to l; now we know only the integer sol'ns (for m and therefore for l) apply to orbital ang. mom.

Sol'n for Θ dependence! So now we have

$$Y_l^m(\theta, \varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi} \Theta_l^m(\theta)$$

and we need to solve for $\Theta_l^m(\theta)$. We can do that by solving $L_+ Y_l^l = 0$ (*)

for Y_l^l and generating the rest of the Y_l^m using L_-.

So in spherical coords becomes (cf. p. 7.15)

$$L_+ e^{il\varphi}\,\Theta_l^l(\theta) = \hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\,\frac{\partial}{\partial\varphi}\right) e^{il\varphi}\,\Theta_l^l(\theta) = 0$$

+ taking the $\frac{\partial}{\partial\varphi}$

$$\hbar e^{i\varphi} e^{il\varphi}\left(\frac{\partial}{\partial\theta} - l\cot\theta\right)\Theta_l^l(\theta) = 0$$

or $$\frac{d}{d\theta}\Theta_l^l = l\cot\theta\,\Theta_l^l$$

Now note $$\cot\theta = \frac{d\sin\theta}{d\theta}\,\frac{1}{\sin\theta}$$

and $$\frac{d\sin^l\theta}{d\theta} = l\sin^{l-1}\theta\cos\theta = l\cot\theta\sin^l\theta$$

$$\Rightarrow l\cot\theta = \frac{1}{\sin^l\theta}\,\frac{d\sin^l\theta}{d\theta} = \frac{d}{d\theta}\ln(\sin^l\theta)$$

So eq'n for Θ becomes

$$\frac{d}{d\theta}\ln(\Theta_l^l(\theta)) = \frac{d}{d\theta}\ln(\sin^l\theta)$$

$$\Rightarrow \Theta_l^l(\theta) = \text{const}\,\sin^l\theta$$

$$\Rightarrow \boxed{Y_l^l(\theta,\varphi) = \frac{A_{ll}}{\sqrt{2\pi}}\sin^l\theta\, e^{il\varphi}}$$

and to get the remaining Y_ℓ^m, use L_-. E.g.

$$Y_\ell^{\ell-1} = L_- Y_\ell^\ell \quad \text{(up to normalization)}$$

$$L_- Y_\ell^\ell = \hbar e^{-i\varphi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)\frac{A_{\ell\ell}}{\sqrt{2\pi}}\sin^\ell\theta\, e^{i\ell\varphi}$$

$$= \text{const}\; e^{-i\varphi}\left[-\ell\sin^{\ell-1}\theta\cos\theta + i\cot\theta\,(i\ell)\sin^\ell\theta\right]e^{i\ell\varphi}$$

$$-2\ell\sin^{\ell-1}\theta\cos\theta$$

$$= A_{\ell,\ell-1}\sin^{\ell-1}\theta\cos\theta\, e^{i(\ell-1)\varphi}$$

etc.

The lowest ℓ & m normalized wave fns are given on p. 373 of the book (next page in these notes),

and more generally,

$$L_\pm |\ell m\rangle = \hbar\left[\ell(\ell+1) - m(m\pm1)\right]^{1/2} |\ell, m\pm1\rangle$$

where $|\ell m\rangle$ and $|\ell, m\pm1\rangle$ are normalized

s.t. $$\langle\ell' m'|\ell m\rangle = \delta_{\ell\ell'}\delta_{mm'}$$

and the overlap integral is over all solid angles.

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TABLE 9.1 The first few normalized spherical harmonics and corresponding associated Legendre functions[a]

$Y_l^m(\theta,\phi) = \left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\theta)e^{im\phi}$	$Y_l^{-l} = \frac{1}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}}\sin^l\theta e^{-il\phi}$
$\int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi\, Y_l^m (Y_{l'}^{m'})^* = \delta_{mm'}\delta_{ll'}$	$Y_l^0 = \sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)$
$P_0 = 1$	$\sum_{m=-l}^{l} \lvert Y_l^m(\theta,\phi)\rvert^2 = \frac{2l+1}{4\pi}$
$P_1^1 = -\sin\theta$	$Y_l^{-m} = (-1)^m (Y_l^m)^*$
$P_1^0 = \cos\theta$	$Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2}$
$P_1^{-1} = \frac{1}{2}\sin\theta$	$Y_1^1 = -\frac{1}{2}\left(\frac{3}{2\pi}\right)^{1/2}\sin\theta\, e^{i\phi}$
$P_2^2 = 3\sin^2\theta$	$Y_1^0 = \frac{1}{2}\left(\frac{3}{\pi}\right)^{1/2}\cos\theta$
$P_2^1 = -3\sin\theta\cos\theta$	$Y_1^{-1} = \frac{1}{2}\left(\frac{3}{2\pi}\right)^{1/2}\sin\theta\, e^{-i\phi}$
$P_2^0 = \frac{1}{2}(3\cos^2\theta - 1)$	$Y_2^2 = \frac{1}{4}\left(\frac{15}{2\pi}\right)^{1/2}\sin^2\theta\, e^{2i\phi}$
$P_2^{-1} = \frac{1}{2}\sin\theta\cos\theta$	$Y_2^1 = -\frac{1}{2}\left(\frac{15}{2\pi}\right)^{1/2}\sin\theta\cos\theta\, e^{i\phi}$
$P_2^{-2} = \frac{1}{8}\sin^2\theta$	$Y_2^0 = \frac{1}{4}\left(\frac{5}{\pi}\right)^{1/2}(3\cos^2\theta - 1)$
$P_3^3 = -15\sin^3\theta$	$Y_2^{-1} = \frac{1}{2}\left(\frac{15}{2\pi}\right)^{1/2}\sin\theta\cos\theta\, e^{-i\phi}$
$P_3^2 = 15\sin^2\theta\cos\theta$	$Y_2^{-2} = \frac{1}{4}\left(\frac{15}{2\pi}\right)^{1/2}\sin^2\theta\, e^{-2i\phi}$
$P_3^1 = -\frac{3}{2}\sin\theta(5\cos^2\theta - 1)$	$Y_3^3 = -\frac{1}{8}\left(\frac{35}{\pi}\right)^{1/2}\sin^3\theta\, e^{3i\phi}$
$P_3^0 = \frac{1}{2}(5\cos^3\theta - 3\cos\theta)$	$Y_3^2 = \frac{1}{4}\left(\frac{105}{2\pi}\right)^{1/2}\sin^2\theta\cos\theta\, e^{2i\phi}$
$P_3^{-1} = \frac{1}{8}\sin\theta(5\cos^2\theta - 1)$	$Y_3^1 = -\frac{1}{8}\left(\frac{21}{\pi}\right)^{1/2}\sin\theta(5\cos^2\theta - 1)e^{i\phi}$
$P_3^{-2} = \frac{1}{8}\sin^2\theta\cos\theta$	$Y_3^0 = \frac{1}{4}\left(\frac{7}{\pi}\right)^{1/2}(5\cos^3\theta - 3\cos\theta)$
$P_3^{-3} = \frac{1}{48}\sin^3\theta$	$Y_3^{-1} = \frac{1}{8}\left(\frac{21}{\pi}\right)^{1/2}\sin\theta(5\cos^2\theta - 1)e^{-i\phi}$
	$Y_3^{-2} = \frac{1}{4}\left(\frac{105}{2\pi}\right)^{1/2}\sin^2\theta\cos\theta\, e^{-2i\phi}$
	$Y_3^{-3} = \frac{1}{8}\left(\frac{35}{\pi}\right)^{1/2}\sin^3\theta\, e^{-3i\phi}$

[a]Defining relations for $P_l(\mu)$ and $P_l^m(\mu)$ are given in Table 9.3. Comparison with other notations for the spherical harmonics and their related functions may be found in D. Park, *Introduction to the Quantum Theory*, 2d ed., McGraw-Hill, New York, 1974.

Ex. Now here's an example that brings in some concepts from earlier in the course. Suppose we have a physical system — a rigid rotator, say — (like a diatomic molecule for instance) in an eigenstate of $L^2 + L_z$ with $l = 1$, $m = 1$ $\Rightarrow Y_1^1(\theta, \varphi)$

Now suppose we measure L_x. We know we can have $L_x = m_x \hbar$, with $m_x = 0, \pm 1$. What is the probability of each?

We have to use the completeness of eigenfunctions. In this case it means that for a given l, the eigenfunctions of L_z with all values of m_l span the space: any function of $\theta + \varphi$ with $L^2 f(\theta,\varphi) = \hbar^2 l(l+1) f(\theta,\varphi)$ can be written as a linear combination of the L_z eigenfns, i.e. of the Y_l^m for this l.

Similarly, any such function can be written as a lin. comb of L_x eigenfunctions.

So to solve this problem, we write Y_1^1 as a linear comb. of L_x eigenfunctions (of which there are 3 for $l=1$); the absolute square of the coefficients gives the prob of finding the corresponding values of m_x.

More specifically, let $\Xi_{m_x}(\theta,\varphi)$ be the eigenfunctions of L_x, with

$$L_x \Xi_{m_x}(\theta,\varphi) = m_x \hbar \, \Xi_{m_x}(\theta,\varphi)$$

There are 3: Ξ_0, Ξ_1, Ξ_{-1}

We can write

$$Y_1^1 = \alpha \Xi_1 + \beta \Xi_0 + \gamma \Xi_{-1}$$

and $|\alpha|^2$ is the prob of $m_x = 1$, etc.

So how do we find this expansion? Write the Ξ's in terms of the Y_1^m. Here's how:

Use $L_x \Xi = \hbar \delta \Xi$

& note $L_x = \frac{1}{2}(L_+ + L_-)$

Then with $\Xi = a Y_1^1 + b Y_1^0 + c Y_1^{-1}$, subst into

We need (see p. 7.19)

$$L_+ Y_1^0 = \sqrt{2}\hbar \, Y_1^1$$
$$L_+ Y_1^{-1} = \sqrt{2}\hbar \, Y_1^0$$
$$L_- Y_1^0 = \sqrt{2}\hbar \, Y_1^{-1}$$
$$L_- Y_1^1 = \sqrt{2}\hbar \, Y_1^0$$

So:
$$\frac{1}{2}(L_+ + L_-)(aY_1^1 + bY_1^0 + cY_1^{-1}) = \hbar\delta(aY_1^1 + bY_1^0 + cY_1^{-1})$$

$$LHS = \tfrac{1}{2}\hbar\sqrt{2}(aY_1^0 + bY_1^{-1} + bY_1^1 + cY_1^0)$$

Combining, we can write this eq'n as (moving everything to LHS + multiplying by $\sqrt{2}$):

$$(-\sqrt{2}\delta a + b)\,Y_1^1(\theta,\varphi)$$
$$+\,(a - \sqrt{2}\delta b + c)\,Y_1^0(\theta,\varphi)$$
$$+\,(b - \sqrt{2}\delta c)\,Y_1^{-1}(\theta,\varphi) = 0$$

Now, this eq'n must be true for all θ and φ, and since Y_1^1, Y_1^0 + Y_1^{-1} are linearly independent, it must be the case that each quantity in parentheses must be equal to zero by itself.

That, in turn means that the following linear system of homogeneous eq'ns must be satisfied:

$$\begin{pmatrix} -\sqrt{2}\delta & 1 & 0 \\ 1 & -\sqrt{2}\delta & 1 \\ 0 & 1 & -\sqrt{2}\delta \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

Which means the determinant of this matrix must vanish

(7.24)

$$det = -\sqrt{2}\,\delta(2\delta^2-1) - (-\sqrt{2}\delta) = -\sqrt{2}\left[\delta(2\delta^2-2)\right] = 0$$

$$\Rightarrow \quad \delta(\delta^2-1) = 0$$

$$\Rightarrow \delta = 0, \pm 1$$

Of course, δ is the same as m_x, so we knew that it must have these values. If we plug bare into the eq'ns we get

$\delta = 0$: $\quad b = 0$, $\quad a + c = 0$ $\quad \Rightarrow \Xi_0 = \frac{1}{\sqrt{2}}(Y_1^1 - Y_1^{-1})$

(+ we've used the normalization condition to get the $\frac{1}{\sqrt{2}}$)

$\delta = \pm 1$: $\quad \left.\begin{aligned} b &= \pm\sqrt{2}\,a \\ &= \pm\sqrt{2}\,c \end{aligned}\right\} \Rightarrow a = c;\ b = \pm\sqrt{2}\,a$

$a + c = \pm\sqrt{2}\,b$

$$\Rightarrow \Xi_{\pm 1} = \frac{1}{2}(Y_1^1 \pm \sqrt{2}\,Y_1^0 + Y_1^{-1})$$

So now we can turn these around and write Y_1^1 in terms of the Ξ's:

$$Y_1^1 = \frac{1}{2}(\Xi_+ + \sqrt{2}\,\Xi_0 + \Xi_-)$$

So finally, in answer to the original question;

prob $\quad m_x = +1 = \frac{1}{4}$

$\quad 0 = \frac{1}{2}$

$\quad -1 = \frac{1}{4}$

<u>Addition of angular momentum</u> (7.25)

We're interested in systems where the total momentum comes fr/ the momenta of the constituents. An example is a multielectron atom, where we're interested in the total orbital angular momentum of two (or more) separate electrons, or the combined spin and orbital angular momentum of a single electron.

For definiteness, we'll think in terms of the orbital angular momenta of two separate electrons,

$$\vec{L}_1 + \vec{L}_2$$

Total ang mom. $\vec{L}$ has

$$L^2 = (\vec{L}_1 + \vec{L}_2)^2 = L_1^2 + L_2^2 + 2\vec{L}_1 \cdot \vec{L}_2$$

$$L_z = L_{1z} + L_{2z}$$

Now each electron has two good quantum numbers associated with its ang. mom (l_1 + m_1 + l_2 + m_2).

So we expect the total angular momentum to be associated with a set of 4 good quantum numbers (and therefore a set of 4 mutually commuting operators)

There are two sets:

<u>Uncoupled representation</u> $|l_1 l_2 m_1 m_2\rangle$

Suppose the state has definite values of $L_{1z}, L_{2z} = m_1\hbar, m_2\hbar$

Then we can still specify l_1 + l_2. For this to be true we must have

$$[L_1^2, L_2^2] = [L_{1z}, L_{2z}] = [L_1^2, L_{1z}] = [L_2^2, L_{2z}] = [L_1^2, L_{2z}]$$

$$= [L_2^2, L_{1z}] = 0$$

(7.26)

We already know L_1^2 commutes w/ L_{1z} + similarly for L_2^2 + L_{2z}. The other relations follow from the fact that the coordinates of the two electrons are independent: $\left[\frac{\partial}{\partial z_1}, z_2\right] = 0$ etc.

So $l_1, l_2, m_1 + m_2$ are a good set with commuting operators. Is that as far as we can go? Can we, for instance also specify l for the whole system? Check commutators

$$[L_{1z}, L^2] = [L_{1z}, L_1^2 + L_2^2 + 2\vec{L}_1 \cdot \vec{L}_2]$$

$$= 2\,[L_{1z}, \vec{L}_1 \cdot \vec{L}_2] = 2[L_{1z}, L_{1x}]L_{2x} + 2[L_{1z}, L_{2y}]L_{2y}$$

$$= 2i\hbar(L_{1y}L_{2x} - L_{1x}L_{2y}) \neq 0$$

So l is not also a good quantum number when $|l_1 l_2 m_1 m_2\rangle$ are specified.

This representation is called because we specify L^2 + L_z for each component of the system separately but we can't say anything about the total angular momentum. (And note that nothing we have said is specific to orbital angular momentum, so the idea applies equally well to spin)

2. Coupled representation

Now suppose we have definite values of L^2 + L_z for the whole system. That gives l and m. We expect there to be two more good quantum no's. They turn out to be l_1 + l_2. [We know they can't be m_1 + m_2 because we already saw e.g. $[L_{1z}, L^2] \neq 0$]

Again, to show this we need to show that (7.27)
all of the corresponding operators commute.
We know $[L_1^2, L_2^2] = 0$ and also $[L_1^2, L_z] = 0$
because L_1^2 commutes w/ both L_{1z} & L_{2z}; $[L_2^2, L_z] = 0$
for similar reasons. What about e.g. $[L_1^2, L^2]$?

$$[L_1^2, L^2] = [L_1^2, L_1^2 + L_2^2 + 2\vec{L}_1 \cdot \vec{L}_2]$$

$$= 2[L_1^2, 2\vec{L}_1 \cdot \vec{L}_2]$$

$$= 2\{[L_1^2, L_{1x}]L_{2x} + [L_1^2, L_{1y}]L_{2y} + [L_1^2, L_{1z}]L_{2z}\}$$

$$= 0 \quad \text{ok}$$

So we can specify $|l\, m\, l_1 l_2\rangle$, but no more. This is called the representation and is relevant to problems where the total angular momentum is important.

So, to summarize:

Uncoupled		Coupled	
L_1^2	l_1	$L^2 = (\vec{L}_1 + \vec{L}_2)^2$	l
L_2^2	l_2	$L_z = L_{1z} + L_{2z}$	m
L_{1z}	m_1	L_1^2	l_1
L_{2z}	m_2	L_2^2	l_2
$\Rightarrow |l_1 l_2 m_1 m_2\rangle$		$|l\, m\, l_1 l_2\rangle$	

Notice that l_1 & l_2 are common to both.

Now since each of these is a complete set, <u>any</u> (7.28)
state can be written as an expansion (superposition) in terms of the eigenstates in either of these representations.

In particular, we can write the coupled eigenstates as a superposition of uncoupled eigenstates. Doing so amounts to answering the question, How do we build up states of definite total angular momentum (coupled) from states with definite values of L_i^2 + L_{iz} for the individual components (uncoupled)?

Expansion in terms of uncoupled rep.: Clebsch-Gordan coefficients

The above implies we can write

$$|l\, m\, l_1 l_2\rangle = \sum_{\substack{m_1 \\ m_1+m_2=m}} \sum_{m_2} C_{m_1 m_2} \, |l_1 l_2 m_1 m_2\rangle$$

Where $C_{m_1 m_2} = \langle l_1 l_2 m_1 m_2 | l\, m\, l_1 l_2 \rangle$

= coefficient

A state w/ total ang. mom. L + L_z is made up of a combination of different contributions from m_1 + m_2 s.t. $m_1 + m_2 = m$

and the expansion implies a probabilistic interpretation:

$|C_{m_1 m_2}|^2$ = prob. that at fixed total L + L_z, a measurement finds $L_{1z} = m_1 \hbar$ and $L_{2z} = m_2 \hbar$

So how do we actually calculate these things?
We can write the expansion explicitly and apply the raising + lowering op's

Let
$$|l\, m\, l_1 l_2\rangle = |1,-1,1,1\rangle$$

So $m=-1$ and we can have
$$m_1 = 0,\quad m_2 = -1 \qquad \text{for } l_1 = l_2 = 1$$
$$\text{or}\quad m_1 = -1,\quad m_2 = 0$$

which means
$$|1,-1,1,1\rangle = C_{0-1}\,|1\,0\rangle_1\,|1\,-1\rangle_2 + C_{-10}\,|1,-1\rangle_1\,|1,0\rangle_2$$

Now if we apply the lowering operator $L_- = L_{1-} + L_{2-}$ to the LHS, we must get 0 because m is already at its minimum value. So

$$L_-|1,-1,1,1\rangle = 0 = C_{0-1}(L_{1-}+L_{2-})\,|1\,0\rangle_1\,|1\,-1\rangle_2$$
$$+\; C_{-10}(L_{1-}+L_{2-})\,|1\,-1\rangle_1\,|1\,0\rangle_2$$

and since $L_\pm|l\,m\rangle = \hbar\left[l(l+1)-m(m\pm 1)\right]^{1/2}|l,m\pm 1\rangle$ (cf p.7.19 in notes or p.379 in book)

$$L_{1-}|1\,0\rangle = \hbar\sqrt{2}\,|1\,-1\rangle_1$$
$$L_{2-}|1\,0\rangle_2 = \hbar\sqrt{2}\,|1\,-1\rangle_2$$

so $$0 = \hbar\sqrt{2}\Big[C_{0-1}\ |1\,{-1}\rangle_1 |1\,{-1}\rangle_2 + C_{-10}\ |1\,{-1}\rangle_1 |1\,{-1}\rangle_2\Big]$$

or $C_{0-1} = -C_{-10}$

Now require that $|1,-1,1,1\rangle$ be normalized $\Rightarrow$ (cf. .)

$$|C_{0-1}|^2 + |C_{-10}|^2 = 1 \Rightarrow 2|C_{0-1}|^2 = 1$$

$$\Rightarrow C_{0-1} = \frac{1}{\sqrt{2}}\ ;\ C_{-10} = -\frac{1}{\sqrt{2}}$$

$\Rightarrow$ Equally probable that $(m_1, m_2) = (0,-1)$ or $(-1,0)$

How do we find other Clebsch-Gordons? We could take the state we know, with the lowest m as in the example above, and apply the raising operator L_+. For example:

$$|1\,0\,1\,1\rangle = C_{1-1}\ |1\,1\rangle_1 |1\,{-1}\rangle_2 + C_{00}\ |1\,0\rangle_1 |1\,0\rangle_2 + C_{-11}\ |1\,{-1}\rangle_1 |1\,1\rangle_2$$

Taking $L_+|1\,{-1}\,1\,1\rangle$ (see eq'n (*)) will give these terms, and it's simply a matter of identifying the coefficients, being careful to get the normalization right.

Coordinate representation Notice that we did not say anything about spherical harmonics $Y_\ell^m(\theta,\varphi)$ in the section on addition of angular momenta, nor did we need to. The beauty of the algebraic method lies in the fact that we don't have to have explicit forms of the eigenstates — i.e. explicit representations in terms of a particular set of coordinates. All we need are the eigenvalue eq'ns.

But if we do want a coordinate representation, we can easily get it; it's equivalent to a projection onto the particular coordinate space.

So, for example, for the eigenvector $|\ell m\rangle$ of $L^2 + L_z$, we can get a (spherical) coordinate representation from

$$\langle \theta, \varphi | \ell m\rangle = Y_\ell^m(\theta,\varphi)$$

and for a composite state $|\ell_1 \ell_2 m_1 m_2\rangle = |\ell_1 m_1\rangle_1 |\ell_2 m_2\rangle_2$

$$\langle \theta_1 \varphi_1 \theta_2 \varphi_2 | \ell_1 \ell_2 m_1 m_2\rangle = Y_{\ell_1}^{m_1}(\theta_1 \varphi_1)\, Y_{\ell_2}^{m_2}(\theta_2 \varphi_2)$$

in the uncoupled representation.

The result in the coupled rep. $|\ell m \ell_1 \ell_2\rangle$ follows from:

$$\langle \theta_1 \varphi_1 \theta_2 \varphi_2 | \ell m \ell_1 \ell_2\rangle = \sum_{\substack{m_1 \\ m_1+m_2=m}} \sum_{m_2} C_{m_1 m_2} \langle \theta_1 \varphi_1 \theta_2 \varphi_2 | \ell_1 \ell_2 m_1 m_2\rangle$$

$$= \sum\sum C_{m_1 m_2}\, Y_{\ell_1}^{m_1}(\theta_1 \varphi_1)\, Y_{\ell_2}^{m_2}(\theta_2 \varphi_2)$$

It's the same Clebsch-Gordon expansion defined above on p. 7.28

Values of total ang. mom. ℓ

But given ℓ_1 + ℓ_2, what are the possible values of total ang. mom. ℓ? Whatever they are, we know they're separated by steps of 1. So we just need to specify ℓ_{max} + ℓ_{min}

ℓ_{max}: We know $L_z = L_{1z} + L_{2z} \Rightarrow m = m_1 + m_2$

So $m_{max} = m_{1max} + m_{2max}$

$= \ell_1 + \ell_2$

But $\ell_{max} = m_{max}$

$$\Rightarrow \boxed{\ell_{max} = \ell_1 + \ell_2}$$

To get ℓ_{min}, use the fact that there are $(2\ell_1+1)(2\ell_2+1)$ independent states for the whole system; this statement is true for any representation. Now for each ℓ, we have $2\ell+1$, so we must have

$$\sum_{\ell_{min}}^{\ell_{max}} (2\ell+1) = \sum_{\ell_{min}}^{\ell_1+\ell_2} (2\ell+1) = (2\ell_1+1)(2\ell_2+1)$$

Now $$\sum_{n_{min}}^{n_{max}} n = \sum_{1}^{n_{max}} n - \sum_{1}^{n_{min}-1} n = \frac{n_{max}(n_{max}+1)}{2} - \frac{(n_{min}-1)n_{min}}{2}$$

So $$\sum_{\ell_{min}}^{\ell_1+\ell_2} (2\ell+1) = 2\left[\frac{(\ell_1+\ell_2)(\ell_1+\ell_2+1) - (\ell_{min}-1)\ell_{min}}{2}\right] + \ell_1+\ell_2-\ell_{min}$$

$$= (2\ell_1+1)(2\ell_2+1)$$

and after some algebra you find

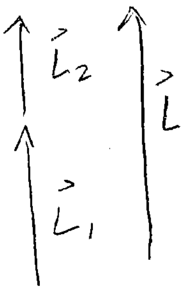
$$l_{min}^2 = (l_1 - l_2)^2$$

or $\boxed{l_{min} = |l_1 - l_2|}$

There is also a graphical way to understand this:

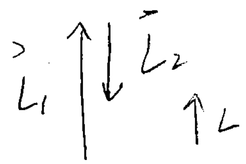
$$\vec{L} = \vec{L}_1 + \vec{L}_2$$

l_{max} has $\vec{L}_1 + \vec{L}_2$ lined up in parallel:



$$L_z = L_{1z\,max} + L_{2z\,max}$$
$$= (l_1 + l_2)\hbar$$

l_{min} has $\vec{L}_1 + \vec{L}_2$ antiparallel



$$L_z = |L_{1z\,max} - L_{2z\,max}|$$

Finally, this is all combined in table 9.5, which gives explicitly the expansions for all nine possible states obtainable from $l_1 = 1$, $l_2 = 1$

Note: In general, if we have $\vec{J} = \vec{J}_1 + \vec{J}_2$, $j_{max} = j_1 + j_2$
$j_{min} = |j_1 - j_2|$

i.e. our result also applies to spin.