Note $S^2 = \vec{S}\cdot\vec{S} = \left(\frac{\hbar}{2}\right)^2 \vec{\sigma}\cdot\vec{\sigma} = \left(\frac{\hbar}{2}\right)^2(\sigma_x^2 + \sigma_y^2 + \sigma_z^2)$

$$= \hbar^2 \underbrace{\tfrac{1}{2}\left(\tfrac{1}{2}+1\right)}_{\frac{3}{4}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and it's degenerate, as we already knew.

Free particle wave functions, with spin

So now we can add spin to the space dependence of a particle's wave fn. Since the spin state and space dependence have nothing to do with each other for a (nonrelativistic) free particle, we can take spin and space variables to commute. So we can have the following sets of commuting op's:

$$\hat{S}^2, \hat{S}_z, \hat{x}, \hat{y}, \hat{z}$$

or $\hat{S}^2, \hat{S}_z, \hat{P}_x, \hat{P}_y, \hat{P}_z$

We already know the free particle Hamiltonian

$$H = \frac{|\vec{P}|^2}{2m}$$

and its eigenfunctions

$$\psi_{\vec{k}}(\vec{r}) = A\, e^{i\vec{k}\cdot\vec{r}}$$

where $A = \frac{1}{(2\pi)^{3/2}}$ for 3 dimensions

To add the spin dependence we simply multiply by the spin vector (we're still taking spin ½): (8.15)

$s = +\frac{1}{2}$ $\quad \psi_+ \equiv \psi_{\vec{k}}(\vec{r})\,\alpha = A e^{i\vec{k}\cdot\vec{r}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$s = -\frac{1}{2}$ $\quad \psi_- \equiv \psi_{\vec{k}}(\vec{r})\,\beta = A e^{i\vec{k}\cdot\vec{r}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

with orthonormality condition $\langle \psi_+ | \psi_- \rangle = 0$ and $\langle \psi_{\pm,k'} | \psi_{\pm k} \rangle = \delta^3(\vec{k} - \vec{k}')$

So we have just constructed simultaneous eigenstates of $\hat{H}$, $\hat{S}^2$ and $\hat{S}_z$, with

$$\hat{H}\psi_\pm = \frac{\hbar^2 |\vec{k}|^2}{2m}\psi_\pm \qquad \hat{S}^2 \psi_\pm = \frac{3\hbar^2}{4}\psi_\pm \qquad \hat{S}_z \psi_\pm = \pm\frac{\hbar}{2}\psi_\pm$$

And notice that we have degeneracy: ψ_+ and ψ_- correspond to the same energy eigenvalue, $\frac{\hbar^2 |\vec{k}|^2}{2m}$.

(This follows from the fact that H does not depend on S_z.)

If we wish to include time dependence, we have just what we'd expect:

$$s = +\tfrac{1}{2} \quad \psi_{\vec{k}}(\vec{r}, t) = A e^{i(\vec{k}\cdot\vec{r} - \omega t)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{with } \omega = \frac{\hbar k^2}{2m}$$

and $\langle \psi_{k'} | \psi_k \rangle = \delta^3(\vec{k} - \vec{k}')$

[$s = -\frac{1}{2}$ follows just as you'd expect.]

Magnetic Moment of an Electron

This is all well and good, but awfully abstract.

Q: Where's the physics, i.e., how does the spin show up in a physically relevant (experimentally measurable) way?

A: Think back to E+M and the fact that magnetic moments respond to magnetic fields. (Remember the picture of mag. mom. as a current loop w/ infinitessimally small radius...)

It turns out that an electron has a magnetic moment proportional to its spin! So if you put an electron in a magnetic field you can (potentially) find out something about its spin. Indeed, that's how Stern + Gerlach showed in 1922 that spin is quantized.

Here's how it all works:

The electron has a magnetic moment

$$\boxed{\vec{\mu} = \frac{e}{mc}\vec{S} = \frac{e\hbar}{2mc}\vec{\sigma} = -\mu_0\vec{\sigma}}$$

where $\mu_0 = \dfrac{|e|\hbar}{2mc} = 0.927 \times 10^{-20}$ erg/gauss

= "Bohr magneton"

(and note e is negative)

8.17

$\Rightarrow$ magnetic moment is in <u>opposite</u> direction from spin.

Now we know from E+M that a magnetic field exerts a torque on a magnetic moment:

$$\vec{\tau} = \vec{\mu} \times \vec{B} \quad \text{for } \vec{B} \text{ uniform, const}$$

so $\vec{\mu}$ wants to be aligned w/ $\vec{B}$.

Well, it takes work to do this rotation, and the associated potential energy is

$$V = -\vec{\mu} \cdot \vec{B}$$

So V is minimized if
$\vec{\mu}$ is parallel to $\vec{B}$

(which occurs if spin is antiparallel to $\vec{B}$)

Now recall $\vec{F} = -\vec{\nabla} V$, which means

$$\vec{F} = \vec{\nabla}(\vec{\mu} \cdot \vec{B})$$

$\Rightarrow$ there's a force on the dipole if $\vec{B}$ is not uniform!

<u>Stern-Gerlach expt.</u>

That's what Stern + Gerlach took advantage of in their experiment.

Using the configuration below, they created a nonuniform $\vec{B}$ field w/ (8.18)

$$\vec{B} \simeq B_z \hat{z}$$

and $$\vec{\nabla} B_z \simeq \frac{\partial B_z}{\partial z} \hat{z}$$

so all the action is in the z direction;

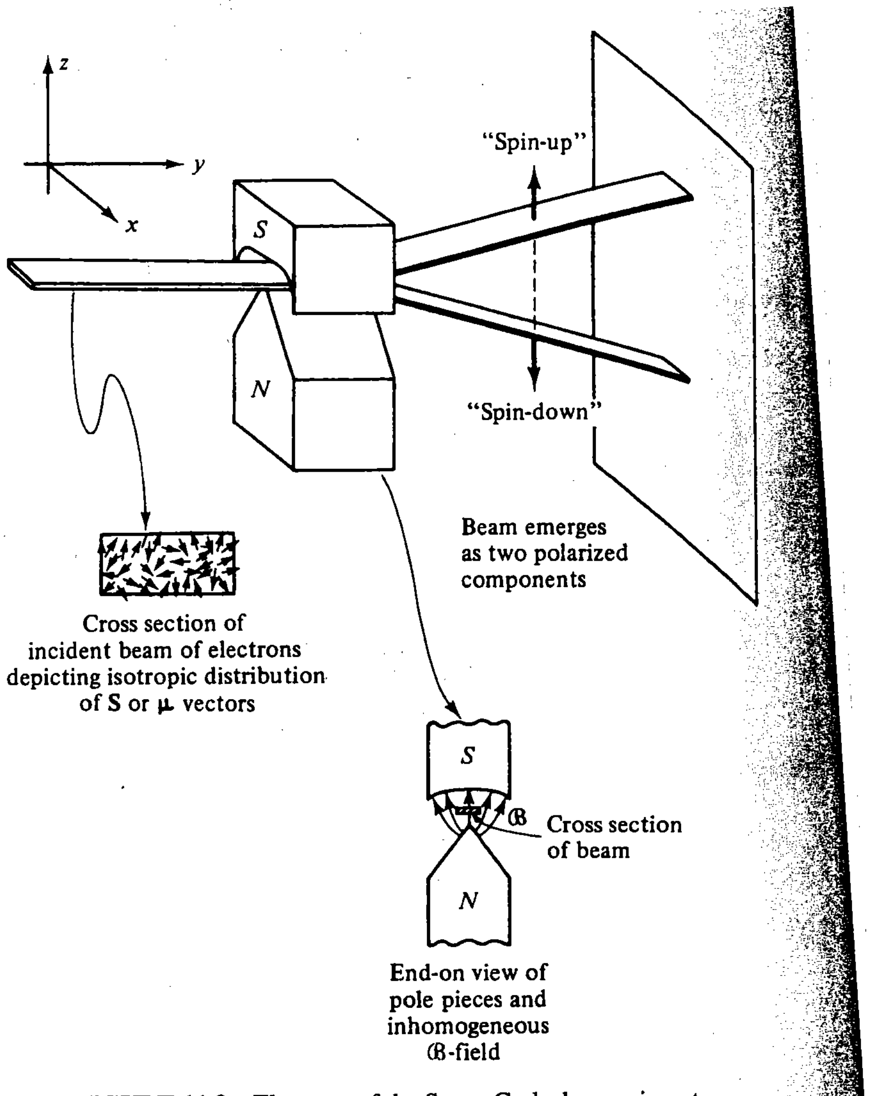


FIGURE 11.9 Elements of the Stern–Gerlach experiment.

They shot a beam of silver atoms, in the y direction through this setup (see figure). Note that a silver atom gets its spin from its outer 5s electron, so for spin purposes this is equivalent to shooting electrons. They did not control the polarization (= spin direction) of the electrons, so the initial beam should be considered to have an isotropic distribution of spin mag moments.

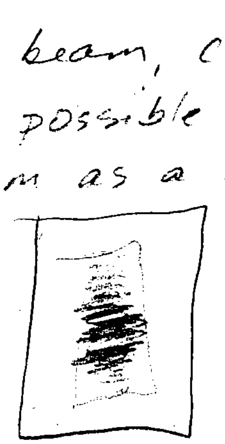
So what happens? The force is

$$\vec{F} = \vec{\nabla}\mu \cdot \vec{B} \simeq \mu_z \frac{\partial B_z}{\partial z}\hat{z}$$

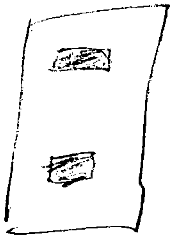
mostly in the z direction $\Rightarrow$ atoms get deflected up or down according to the sign of μ_z ($\frac{\partial B_z}{\partial z}$ is known and controlled by the experimenters.)

So if we measure the deflection, that gives the force, which gives the magnetic moment, which gives the spin.

Now, with an isotropic beam, classically we'd expect that we'd have all possible values of μ_z, so we'd see the deflected beam as a continuum on a detection screen:



In fact, Stern and Gerlach saw instead (8.20)
two separate pieces:



$\Rightarrow$ μ_z and hence S_z can take on only two equal and opposite values, which is exactly what we get if $s = \frac{1}{2}$!

Now the two outgoing beams are, according to the postulates of QM, respectively eigenstates of $S_z = +\frac{1}{2}$ and $S_z = -\frac{1}{2}$.

Now what if we started w/ a polarized incident beam, w/all spins in the +x direction, say? What happens? The force is still in the z direction, dominated by the z component of the magnetic moments, so we still get discrete up or down deflection. What is the probability of each?

We just need the wave function in the S_z basis.

You will show in the HW that the eigenvector for $S_x = +\frac{1}{2}$ is

$$\alpha_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \underset{\uparrow \, S_z = +\frac{1}{2}}{\frac{1}{\sqrt{2}}\alpha_{(z)}} + \underset{\uparrow \, S_z = -\frac{1}{2}}{\frac{1}{\sqrt{2}}\beta_{(z)}}$$

The probabilities follow directly:

$$P(s_z = +\tfrac{1}{2}) = \left|\tfrac{1}{\sqrt{2}}\right|^2 = \tfrac{1}{2}$$

$$P(s_z = -\tfrac{1}{2}) = \tfrac{1}{2}$$

Incidentally, the wave function for the incident beam was

$$\psi = A e^{i(ky-\omega t)} \alpha_x = \frac{A}{\sqrt{2}} e^{i(ky-\omega t)} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Addition of two spins

Just as with addition of orbital angular momentum, addition of spin can be described in two representations:

uncoupled $|s_1 \, s_2 \, m_{s_1} \, m_{s_2}\rangle$ $S_1^2, S_2^2, S_{1z}, S_{2z}$ eigenstates

$= |\tfrac{1}{2}, \tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle$

coupled $|s, m_s, s_1, s_2\rangle$ $S^2\, S_z, S_1^2, S_2^2$

The uncoupled rep is straightforward to construct (+ let's introduce a shorthand notation)

$\uparrow\uparrow$ $\xi_1 = \alpha(1)\alpha(2)$ $s_1 = \tfrac{1}{2}, s_2 = \tfrac{1}{2}, m_{s_1} = +\tfrac{1}{2}, m_{s_2} = -\tfrac{1}{2}$

$\downarrow\downarrow$ $\xi_2 = \beta(1)\beta(2)$ $-\tfrac{1}{2}$ $-\tfrac{1}{2}$

$\uparrow\downarrow \quad \xi_3 = \alpha(1)\,\beta(2) \qquad m_{s_1} = +\frac{1}{2} \quad m_{s_2} = -\frac{1}{2}$ (8.22)

$\downarrow\uparrow \quad \xi_4 = \beta(1)\,\alpha(2) \qquad -\frac{1}{2} \qquad +\frac{1}{2}$

These are related to the states in the coupled rep. by Clebsch-Gordan coeff's. Let's work it all out.

When we combine spin-½ + spin-½ we can have values of $S = 0, 1$ from $s = |s_1 - s_2|, \ldots s_1 + s_2$

So our eigenstates in the coupled rep are

$S = 0: \qquad |S\, m_s\, s_1\, s_2\rangle = |0\,0\,\frac{1}{2}\,\frac{1}{2}\rangle$

$S = 1$

$m_s = +1 \qquad |1\,1\,\frac{1}{2}\,\frac{1}{2}\rangle$

$m_s = 0 \qquad |1\,0\,\frac{1}{2}\,\frac{1}{2}\rangle$

$m_s = -1 \qquad |1\,-1\,\frac{1}{2}\,\frac{1}{2}\rangle$

and our job is to figure out how these 4 states come from $\xi_1, \xi_2, \xi_3, \xi_4$.

First note that each of the ξ's is already an eigenstate of $S_z = S_{1z} + S_{2z}$. We have $m_s = m_{s_1} + m_{s_2}$

and $\longrightarrow$

$$S_z \xi_1 = (S_{z_1} + S_{z_2})\alpha(1)\alpha(2) = \left(\frac{\hbar}{2} + \frac{\hbar}{2}\right)\alpha(1)\alpha(2) \Rightarrow m_s = +1$$

$$S_z \xi_2 = (S_{z_1} + S_{z_2})\beta(1)\beta(2) = \left(-\frac{\hbar}{2} - \frac{\hbar}{2}\right)\beta(1)\beta(2) \Rightarrow m_s = -1$$

$$S_z \xi_3 = (S_{z_1} + S_{z_2})\alpha(1)\beta(2) = \left(\frac{\hbar}{2} - \frac{\hbar}{2}\right)\alpha(1)\beta(2) \Rightarrow m_s = 0$$

$$S_z \xi_3 = (S_{z_1} + S_{z_2})\beta(1)\alpha(2) = \left(-\frac{\hbar}{2} + \frac{\hbar}{2}\right)\beta(1)\alpha(2) \Rightarrow m_s = 0$$

It seems obvious, then, that

$$\xi_1 \longleftrightarrow s = 1,\ m_s = +1$$

$$\xi_2 \longleftrightarrow s = 1,\ m_s = -1$$

but we have to demonstrate explicitly that these are eigenstates of S^2 for $s = 1$ (We'll deal with $m_s = 0$ below; it's more complicated.)

We need to show

$$S^2 \xi_1 = \hbar^2 (1)(1+1)\xi_1 = 2\hbar^2 \xi_1 \qquad (*)$$

To do this, write

$$S^2 = (\vec{S}_1 + \vec{S}_2)^2 = S_1^2 + S_2^2 + 2\vec{S}_1 \cdot \vec{S}_2$$

and we can express $\vec{S}_1 \cdot \vec{S}_2$ in terms of $S_\pm, S_z$:

$$\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}(S_{1+}S_{2-} + S_{1-}S_{2+}) + S_{1z}S_{2z}$$

Back to eq'n (*) on previous page:

$$S^2 \xi_1 = \left(S_1^2 + S_2^2 + \underbrace{\cancel{S_{1+}}S_{2-}}_{0 \text{ since } S_{1+}\alpha(1)=0} + \overbrace{S_{1-}\cancel{S_{2+}}}^{0 \text{ since } S_{2+}\alpha(2)=0} + 2S_{1z}S_{2z}\right)\alpha(1)\alpha(2)$$

$$= \hbar^2 \underbrace{\left(\frac{3}{4} + \frac{3}{4} + 2\frac{1}{2}\frac{1}{2}\right)}_{2} \alpha(1)\alpha(2)$$

$$= 2\hbar^2 \xi_1$$

$\Rightarrow$ ξ_1 is indeed an eigenstate of S^2 for $S=1$,

similarly, ξ_2 also has $S^2\xi_2 = 2\hbar^2\xi_2$

Now for $m_s = 0$ we're going to deviate a bit from the book. We can find the eigenstate for $s=1$, $m_s=0$ simply by applying the lowering operator $S_- = S_{1-} + S_{2-}$ to ξ_1:

$$S_- |1 1 \tfrac{1}{2} \tfrac{1}{2}\rangle = \hbar\sqrt{1(1+1) - 1(1-1)}\, |1 0 \tfrac{1}{2}\tfrac{1}{2}\rangle \quad \text{(cf p. 8.12)}$$

$$= \sqrt{2}\,\hbar\, |1 0 \tfrac{1}{2}\tfrac{1}{2}\rangle$$

So

$$|1\,0\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}\hbar} S_- |1\,1\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle$$

$$= \frac{1}{\sqrt{2}\hbar}(S_{1-} + S_{2-})\xi_1 = \frac{1}{\sqrt{2}\hbar}(S_{1-} + S_{2-})\alpha(1)\alpha(2)$$

$$= \frac{1}{\sqrt{2}\hbar}\hbar\Big[\underbrace{\sqrt{\tfrac{1}{2}(\tfrac{3}{2}) - \tfrac{1}{2}(\tfrac{1}{2}-1)}}_{1}\,\beta(1)\alpha(2) + \sqrt{1}\,\alpha(1)\beta(2)\Big]$$

$$\Rightarrow |1\,0\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}}\big[\beta(1)\alpha(2) + \alpha(1)\beta(2)\big]$$

$$= \frac{1}{\sqrt{2}}(\xi_3 + \xi_4) \qquad s = 1,\ m_s = 0$$

So now we have all three $s=1$ states, with one remaining to determine: $s=0$, $m_s=0$.

But we know this state $|0\,0\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle$ must be a linear combination of ξ_3 and ξ_4, and furthermore that it must be <u>orthogonal</u> to $|1\,0\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle$. You can confirm that the correct state is

$$|1\,0\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}}(\xi_3 - \xi_4)$$

$$= \frac{1}{\sqrt{2}}\big(\alpha(1)\beta(2) - \beta(1)\alpha(2)\big)$$

(and note that this state is uniquely determined only up to a complex phase.

So we're done. Collecting the results, we have (8.26)
(cf table 11.3 p. 539) (and assuming $s_1 = s_2 = \frac{1}{2}$)

	s	m_s	wavefunction
$\uparrow\uparrow$	1	1	$\alpha(1)\alpha(2)$
$\uparrow\downarrow + \downarrow\uparrow$	1	0	$\frac{1}{\sqrt{2}}\left[\alpha(1)\beta(2) + \beta(1)\alpha(2)\right]$
$\downarrow\downarrow$	1	-1	$\beta(1)\beta(2)$
$\uparrow\downarrow - \downarrow\uparrow$	0	0	$\frac{1}{\sqrt{2}}\left[\alpha(1)\beta(2) - \beta(1)\alpha(2)\right]$

On a final note, notice that all of the $s=1$ states (in particular $s=1$, $m_s=0$) are symmetric under the exchange of particle labels, whereas $s=0, m_s=0$ is antisymmetric. The symmetry of $s=1, m_s=0$ follows from the symmetry of the $s=1$ $m_s=1$ state $\alpha(1)\alpha(2)$. We obtained $|1\,0\,\frac{1}{2}\frac{1}{2}\rangle$ from $\alpha(1)\alpha(2)$ by applying $S_- = S_{1-} + S_{2-}$, which is itself symmetric under particle exchange.