II. Electromagnetic Interactions

"Tea time's over – back on your heads."

Quantum Electrodynamics is the best (and best-beloved) theory we have. The gauge group $U(1)$ is abelian, the coupling constant is small, and life is relatively simple.

Eventually we'll calculate the cross section for $e^+e^- \to \mu^+\mu^-$ in QED, but first, a review of relativistic kinematics, 4-vector notation, + gauge invariance in classical E+M.

For more details: Jackson Ch. 11 or favorite ref. on kinematics

(A) Relativistic Kinematics, 4-vectors, etc

Note: This is only meant to be a review; see ~~Tipton's~~ notes or Jackson for more thorough discussion.

Position 4-vector $\quad x^\mu = (t, \underbrace{x, y, z}_{\text{space comp.}}) = (t, \vec{x})$

t: 0^{th} or time component

Momentum 4-vector: $\quad p^\mu = (E, \vec{p})$, E = energy

$E = \gamma m$

$|\vec{p}| = \gamma m v \; (= \gamma m \beta)$

Generic 4-vector $\quad a^\mu = (a^0, \vec{a})$

Square of 4-vector $\quad a \cdot a = a^2 = (a^0)^2 - \vec{a}\cdot\vec{a}$ = scalar product

is Lorentz invariant

For a particle of mass m, with momentum P,

$$P^2 = E^2 - |\vec{p}|^2 = m^2$$

en-mom cons:
ex. particle decay

real vs. virtual

More notation:

$$x^\mu = (x^0, \vec{x}) \equiv \underline{\text{contravariant}} \text{ vector}$$

$$x_\mu = (x^0, -\vec{x}) \equiv \underline{\text{covariant}} \text{ vector}$$

$\uparrow$

HW!
assume on shell, then look at consequences of 4-mom conservation

Scalar product is $x^\mu x_\mu = (x^0)^2 - \vec{x}^2$

(Summation convention; Greek indices for 4-vectors, repeated indices summed.)

Cov. + contrav. vectors are connected via **metric tensor**

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

$x_\mu = g_{\mu\nu} x^\nu$ + vice versa

[$\Rightarrow$ scalar product $x^2 = g_{\mu\nu} x^\mu x^\nu$]

N.B. Perkins does not use this convention! He uses $P = (iE, \vec{p})$ so $P^2 = P_0^2 + |\vec{p}|^2 = -E^2 + |\vec{p}|^2 = -m^2$.
Be careful not to get confused.

Note that the metric contracted with itself gives

$$g^{\mu\alpha} g_{\alpha\nu} = \delta^{\mu}_{\nu} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \text{ Kroneker } \delta$$

Tensors + transf. properties; We need one more technical detail so that we get the derivative operators defined correctly, because they're not exactly what you might guess.

Technically, cov. vs. contrav. vectors are defined acc'ding to their properties under Lorentz transformations:

$$\text{contrav}: \quad a'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} a^{\nu}$$

This guarantees invariance of scalar prod.

$$\text{cov.}: \quad a'_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} a_{\nu}$$

Usually the difference comes down to a minus sign in the space part of a_μ.

What about derivative operators?

Is $\frac{\partial}{\partial x^\mu}$ contravariant like x^μ or covariant like x_μ?

$$\text{Chain rule}: \quad \frac{\partial}{\partial x'^{\mu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{\partial}{\partial x^{\nu}}$$

$\Rightarrow$ covariant!

So, we have

$$\boxed{\begin{aligned}\partial_\mu &\equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla}\right)\\ \partial^\mu &\equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial x^0}, -\vec{\nabla}\right)\end{aligned}}$$

So watch out for signs! Derivatives + ordinary vectors work slightly differently; $x_\mu = (x^0, -\vec{x})$ but $\partial_\mu = \left(\frac{\partial}{\partial x^0}, \vec{\nabla}\right)$.

This means that the 4-divergence is

$$\boxed{\partial^\mu a_\mu = \partial_\mu a^\mu = \frac{\partial a^0}{\partial x^0} + \vec{\nabla}\cdot\vec{a},}$$

and it's Lorentz invariant. Then there's the 4-Laplacian, or d'Alembertian

$$\boxed{\Box \equiv \partial^\mu \partial_\mu = \frac{\partial^2}{\partial x^{0\,2}} - \nabla^2,}$$

which is invariant + is the familiar wave op. $\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$

Note: The transformations on p. 2.3 generalize to higher rank tensors, e.g.

$$F'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x'^\nu}{\partial x^\sigma} F^{\rho\sigma}$$

+ similarly for $F_{\mu\nu}$, etc

Q: What is $F'^{\ \nu}_{\mu}$? Transform each index separately

Comments on Lorentz invariance

The volume element in 4-space (d^4x) is invariant:

$$d^4x' = (\underset{\substack{\uparrow \\ \text{transf. matrix}}}{\det A})\, d^4x = (+1)\, d^4x$$

The Jacobian of the transformation is just the determinant of the transformation matrix A that operates on 4-vectors. It has $\det A = +1$ (see Jackson for details).

This is true in momentum space: d^4p, d^4k etc are inv.

Not only that, but so is the on-mass-shell condition requiring $p^2 = m^2$. I.e., $\delta(p^2 - m^2)$ is invariant (Manifestly — it only involves scalars). ~~It~~ follows that

$$\delta(p^2 - m^2)\, d^4p$$

is also invariant. But this is just what you would use to integrate over all momentum space for a particle: you'd use d^4p but subject to $p^2 = m^2$. Let's integrate over the time component:

$$\int dp^0 d^3p\, \underbrace{\delta(p^2 - m^2)}$$

$$\delta(p^{0\,2} - |\vec{p}|^2 - m^2) = \delta(p^{0\,2} - E^2)$$

$$= \frac{1}{2E}\left[\delta(p^0 + E) + \delta(p^0 - E)\right]$$

Why keep the 2? Can't have $p^0 < 0$

$$= \frac{d^3p}{2E} \Rightarrow \boxed{\frac{d^3p}{E} \text{ also invariant}}$$

B Gauge invariance in classical E+M

Recall Maxwell's eq'ns:

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$$

$$\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = 4\pi\vec{J}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$$

The charge density ρ and current density $\vec{J}$ satisfy the continuity eq'n

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0,$$

which just expresses conservation of charge, + follows from Maxwell's. The last 2 Maxwell's imply that $\vec{E}$ + $\vec{B}$ can be written in terms of scalar + vector potentials:

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$$

Then the 1st 2 Maxwell's determine ϕ + $\vec{A}$, but not uniquely. Any change in $\vec{A}$, ϕ that leaves $\vec{E}$, $\vec{B}$ unchanged is allowed. Such a change is, we know, a gauge transformation. In general, for any function $\Lambda(\vec{x},t)$,

$$\vec{A} \to \vec{A} + \vec{\nabla}\Lambda$$

$$+\ \phi \to \phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t}$$

leaves $\vec{E}$ + $\vec{B}$ unchanged.

We can use the gauge freedom to put constraints on the potentials acc'ding to the problem at hand, to make it easier to solve. E.g.,

Lorentz gauge: $\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0$ covariant; very useful

Coulomb gauge: $\vec{\nabla}\cdot\vec{A} = 0$ ϕ just like static potential

Covariant notation

Now we reexpress this in covariant notation that will carry over to quantum field theory.

Current: we write the current as a 4-vector

$$\boxed{j^\mu = (\rho, \vec{J})}$$

in which case the continuity eq'n $\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$ becomes

$$\boxed{\partial_\mu j^\mu = 0}$$

and the current is said to be conserved. (This def. is general.)

Aside: We know $\vec{J}$ is a 3-vector + therefore a candidate for the space part of a 4-vector. That ρ transforms like the time component follows from invariance of electric charge. (Electron has same chg no matter what its motion.) $\rightarrow$

element of chg $= \rho d^3x$ invariant $\Rightarrow \rho$ transforms like $dx^0 \Rightarrow$ chg density is time part of 4-vector.

Vector + scalar potential: also form 4-vector

$$\boxed{A^\mu = (\phi, \vec{A})}$$

Field strength tensor: $\vec{E}$ + $\vec{B}$ are not parts of 4-vectors. Instead, they come from the field strength tensor

$$\boxed{F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu}$$

Each nonzero component of this antisymmetric tensor is a component of $\vec{E}$ or $\vec{B}$, up to a sign:

$$F^{0i} = -E^i$$

$$F^{ij} = -\epsilon^{ijk} B^k$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

N.B. The vector potential A^μ, and $F^{\mu\nu}$ in terms of it, will be fundamental in what's to come in QED. The $\vec{E}$ + $\vec{B}$ fields we won't care about much.

Maxwell's in terms of $F^{\mu\nu}$

① Inhomogeneous pair (the ones w/ sources)

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$$

$$\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}$$

can be written

$$\partial^i F^{oi} = 4\pi j^o$$

$$\partial_m F^{mi} + \partial_0 F^{oi} = \frac{4\pi}{c} j^i \qquad (\vec{\nabla}\times\vec{B})_i = \partial_m F^{im}$$

combining, + noting diagonal F's give 0, we have

$$\boxed{\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu}$$

② Homogeneous pair

$$\left.\begin{array}{l}\vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0\end{array}\right\}$$

writing in terms of components of F, we get cyclic permutations, + we can write

$$\boxed{\partial^\rho F^{\mu\nu} + \partial^\nu F^{\rho\mu} + \partial^\mu F^{\nu\rho} = 0}$$

Aside: Just as follow when we define $\vec{E}+\vec{B}$ in terms of potentials, so does. We have

$$\partial^\rho(\partial^\mu A^\nu - \partial^\nu A^\mu) + \partial^\nu(\partial^\rho A^\mu - \partial^\mu A^\rho) + \partial^\mu(\partial^\nu A^\rho - \partial^\rho A^\nu)$$

$$= 0 \quad \text{automatically}$$

Even more compact form comes fr/using dual tensor.

In analogy w/ ϵ_{ijk}, define completely antisymm. tensor of rank 4:

$$\epsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 & \mu\nu\rho\sigma = \text{even perm. of } 0123 \\ -1 & \text{odd perm.} \\ 0 & \text{any pair equal} \end{cases}$$

$\epsilon_{\mu\nu\rho\sigma} = -\epsilon^{\mu\nu\rho\sigma}$ which you can see from lowering all 4 indices; get $g_{00}g_{11}g_{22}g_{33} = -1$

Dual field strength tensor $\mathcal{F}^{\mu\nu}$ is defined as

$$\mathcal{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

also written F^* or $^*F$

Like $F^{\mu\nu}$, $\mathcal{F}^{\mu\nu}$ is a totally antisymm. tensor of rank 2

$$F \to \mathcal{F} \Leftrightarrow \vec{E} \to \vec{B} \text{ and } \vec{B} \to -\vec{E}$$

Hom. Maxwell's become $\boxed{\partial_\mu \mathcal{F}^{\mu\nu} = 0}$

So concise statement of Maxwell's eq'ns:

$$\boxed{\begin{aligned} \partial_\mu F^{\mu\nu} &= \frac{4\pi}{c} j^\nu \\ \partial_\mu \mathcal{F}^{\mu\nu} &= 0 \end{aligned}}$$

Gauge invariance

Recall $\vec{A} \to \vec{A} + \vec{\nabla}\Lambda$
$\phi \to \phi - \partial\Lambda/\partial t$ } leave $\vec{E}, \vec{B}$ unchanged

With $A^\mu = (\phi, \vec{A})$, we can write this [recalling that $\partial^\mu = (\frac{\partial}{\partial t}, -\vec{\nabla})$] as

$$\boxed{A^\mu \to A^\mu - \partial^\mu \Lambda}$$

Note the great feature of the field strength tensor $F^{\mu\nu}$ — that it's invariant under gauge ~~transf~~:

$$F^{\mu\nu} \to \partial^\mu A^\nu - \partial^\mu\partial^\nu\Lambda - \partial^\nu A^\mu + \partial^\nu\partial^\mu\Lambda$$
$$= \partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu}$$

Hence **Maxwell's eq'ns in terms of field strength tensor are manifestly gauge invariant.**

Eq'n of motion for A^μ:

$$\boxed{\Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = j^\mu}$$

Often take $\partial_\nu A^\nu = 0$ (Lorentz gauge)
$\Rightarrow$ each component A^μ satisfies K-G eq'n

U(1) Gauge Invariance in Non-Rel. QM *

i) can see even in non-relativistic quantum mechanics that <u>requiring invariance under local U(1) gauge transformations automatically gives EM interactions.</u>

I) what "local U(1) gauge transformation" means.

[Actually, "gauge" is somewhat redundant; gauge is just a generic name for these transformations of fields. The name gauge, which <u>was</u> originally used in the sense of scale of measurement, is left over from when Weyl tried to unify gravity + electromagnetism, under a principle of changes of scale.]

) — "transformation": what we're transforming is particle wavefunctions + eventually the field operators corresp. to the particles. This includes the EM vector potential A^μ, whose status gets elevated to the field of the photon, as we'll see.

$$\psi \rightarrow \psi' = \underset{\uparrow}{U}\psi$$

operator/matrix that transforms w.f. ψ

* Ref.'s: Quigg, chap. 3
Aitchison + Hey, Chap. 2

— "$U(1)$": Resurrect your group theory (cf. Tipton's notes) and recall that

$U(N)$ = group whose elements give unitary transformations in N dimensions.

unitary $\Leftrightarrow$ $U^{+}U = 1$, where 1 is the $N \times N$ identity matrix, for $U \in U(N)$

In 1-dim, $U^{+}U = U^{*}U = 1$

1-dim matrix = number

$\Rightarrow \boxed{U = e^{i\alpha}}$, where α is real

$\Rightarrow \boxed{U(1) \Leftrightarrow \text{changes of phase}}$: $\psi \to \psi' = e^{i\alpha}\psi$

— "local": As opposed to global.

A global transf. is the same everywhere in spacetime; e.g. $\psi \to e^{i\alpha}\psi$ where α = const

A local transf. is space-time dependent, & is not the same everwhere, e.g.

$$\psi \to e^{i\alpha(x,t)}\psi,$$

where α depends on x & t

② Invariance in non-rel QM

– Global $U(1)$

Note that QM is automatically invariant under global phase transformations of the wave function ψ. The Schrödinger eq'n

$$i\frac{\partial\psi(\vec{x},t)}{\partial t} = \frac{1}{2m}(-i\vec{\nabla})^2\psi(\vec{x},t)$$

becomes, w/ $\psi' \rightarrow e^{i\alpha}\psi(x,t) \equiv \psi'$

$$i\frac{\partial\psi'(\vec{x},t)}{\partial t} = \frac{1}{2m}(-i\vec{\nabla})^2\psi'(\vec{x},t)$$

$$\Leftrightarrow e^{i\alpha}\left[\text{original SE}\right]$$

So if ψ satisfies the free S.E., so does ψ'.

Also, since all observables involve

$$\psi^*(\ldots)\psi,$$

they too are invariant. This just says that the absolute phase of a wavefunction is neither physical nor measurable, a fact you already took advantage of last semester e.g. in choosing phase conventions to discuss $C\!\!\!/P$.

(**N.B.** Phase differences are observable, of course, through interference effects.)

— Local $U(1)$

Let $\boxed{U = e^{iq\Lambda(\vec{x},t)}}$, where now the phase depends on space + time, + we've pulled out a factor of q (guess what it ends up being) for future convenience. So

$$\psi \to \psi'(\vec{x},t) = U(\vec{x},t)\,\psi(\vec{x},t) = e^{iq\Lambda(\vec{x},t)}\,\psi(\vec{x},t)$$

Schr. eq'n becomes

$$e^{iq\Lambda}\left[\left(i\frac{\partial}{\partial t} - q\frac{\partial\Lambda}{\partial t}\right)\psi(\vec{x},t) = \frac{1}{2m}\left(-i\vec{\nabla} + q\vec{\nabla}\Lambda\right)^2\psi(\vec{x},t)\right]$$

$\Rightarrow$ The SE is <u>not</u> invariant under this transf.

However, if there were another field around, described by $\vec{A}$ + ϕ, that coupled to ψ so that

$$\left(i\frac{\partial}{\partial t} - q\phi\right)\psi(\vec{x},t) = \frac{1}{2m}\left(-i\vec{\nabla} - q\vec{A}\right)^2\psi(\vec{x},t),$$

and ϕ + $\vec{A}$ transformed according to

$$\vec{A} \to \vec{A} + \vec{\nabla}\Lambda$$

$$\phi \to \phi - \frac{\partial\Lambda}{\partial t}$$

Then the offending terms cancel and invariance is restored. We recognize these as exactly the classical EM gauge transformations

... EM interactions!

Comments

- Note that U(1) is abelian, i.e., its elements commute with one another and so order does not matter when you make multiple transformations:

$$U_\alpha U_\beta = U_\beta U_\alpha \quad e^{i\alpha} e^{i\beta} = e^{i(\alpha+\beta)} = e^{i\beta} e^{i\alpha}$$

This is true for local and global $U(1)$.

- The vector and scalar potential $\vec{A}$ & ϕ, which were only mathematical conveniences in classical E&M now take on a life of their own and have **physical meaning** in their own right. Their relation to phases of wave fns, as applied above, leads to real effects. For a more full discussion see Aitchison & Hey, pp. 57–60, or Quigg, pp. 43–45. The point is that the wave fn. ψ in the presence of $\vec{A}$ can be related to the free w.f. ψ_0:

$$\psi(\vec{x},t) = e^{iq\int d\vec{x}\cdot\vec{A}}\, \psi_0(x,t)$$

The phase gets shifted by the presence of the potential. This gives rise, among other things, to the Aharonov-Bohm Effect, in which interference makes these phase shift effects observable, even when particles feel no force. (In the real thing, we get the covariant generalization of $\int d\vec{x}\cdot\vec{A}$, i.e. $\int dx^\mu A_\mu$.)

- The way we threw in the $\vec{A}$ + ϕ terms is not as arbitrary as it looks. Note that what we did was

$$\vec{\nabla} \to \vec{\nabla} - iq\vec{A} \equiv \vec{D}$$

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + iq\phi \equiv D^0$$

In covariant form, this is simply (recall $\partial^\mu = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)$)

$$\boxed{\partial^\mu \to D^\mu \equiv \partial^\mu + iqA^\mu}$$

D^μ is called the covariant derivative. This generalizes to relativistic quantum field theory, and to non-abelian gauge transformations:

$$\boxed{\text{Gauge fields couple to matter via covariant derivatives.}}$$

Notice that the coupling constant - in this case the electric charge - is built in. The transformation properties of the matter fields (and the gauge fields themselves) are determined by the gauge group and the rep. of the matter fields. Will see how all this works in more detail for a non-abelian example (SU(2)) later on.

But still, why the covariant derivative? Sure, it gives us a rule, but isn't the rule arbitrary? Nope! Unfortunately, the explanation is beyond the scope of this course. Suffice it to say that there is deep mathematical (in particular, geometrical) significance to the covariant derivative, and it all comes out of a more mathematical treatment of gauge theory, along with buzzwords like "cotangent bundle." It's all there and it's all beautiful. Your classmates ~~Teo Turgut + Bob Henderson~~ are experts on this, and I refer you to them for further discussion.*

As with much of the theory behind what we're doing, I'm mostly presenting the rules that tell us where things fit in + how to do calculations, and leaving the theoretical underpinnings to field theory courses.

<u>Note on textbooks</u>: I mentioned early on that Quigg would be used a lot in preparing lectures, mainly because it's where I learned much of this stuff. I'm finding that Aitchison + Hey is at a more appropriate level ~~for~~ this course (it's less advanced) and I'm using it a lot.

* Note that this point about the origin of the covariant derivative + the previous one (p. 2.16) about phases are deeply related.