Review: Klein-Gordon + Dirac Eq'ns

We're working our way up to calculating the cross section for $e^+e^- \to \mu^+\mu^-$ in QED. First we have to review relativistic wave eq'ns for spin-0 (Klein-Gordon) and spin-½ particles (Dirac). This is just to remind you of notation! For a more comprehensive review consult your favorite book e.g.

Mandl + Shaw, QFT (early chap's, Appendix A)
↳ my favorite

Aitchison + Hey, Gauge Th. in Part. Phys. (Ch. 3; Sec 6.1-2; also ch. 4)
chap 6 ↓
↳ used for these notes

Bjorken + Drell, Relativistic Quantum Mechanics
↳ classic.

etc.

① Klein-Gordon eq'n

The K-G eq'n is for the wave functions $\phi(x)$ of spin-0 particles. The point is to have an eq'n that satisfies

$$E^2 = |\vec{p}|^2 + m^2$$

Substituting for the usual operators

$$E \to i\frac{\partial}{\partial t}$$

$$p \to -i\vec{\nabla}$$

→

gives

$$-\frac{\partial^2\phi}{\partial t^2} = \left(-\nabla^2 + m^2\right)\phi$$

or, in covariant notation (recall $\Box \equiv \partial^\mu \partial_\mu$)

$$\boxed{\left(\Box + m^2\right)\phi(x) = 0} \quad \text{Klein-Gordon eq'n}$$

Note that $p^\mu \Leftrightarrow i\partial^\mu$ so $(p^2 - m^2)\phi = 0$ as required.

Comments

- $\phi(x)$ will become a quantum field below. It will still satisfy the K-G eq'n, which will be the eq'n of motion that comes from the Lagrangian density.
- This is a free wave eq'n – no interactions. It has plane wave sol'ns $\phi(x) \sim e^{-ip\cdot x}$.

There were originally all kinds of problems associated w/ the fact that $E = \pm\sqrt{|\vec{p}|^2 + m^2}$ could be negative, and as a consequence, the probability density was not positive definite – not a good feature for a probability density. Suffice it to say that the negative energy sol'ns are reinterpreted as antiparticles and there's no problem in field theory. See Aitchison + Hey for a nice discussion (chap. 3).

— The K-G eq'n is satisfied by each component of the photon field in Lorentz gauge. (see below). Spin-1 fields are described by four-vectors, such as A^μ for the photon. Recall that the classical eq'n of motion for A^μ is (w/no sources)

$$\Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = 0$$

In Lorentz gauge, $\partial_\nu A^\nu = 0$, and each component of A satisfies the massless K-G eq'n.

— ϕ must be complex to describe a charged scalar field.

② Dirac Eq'n

Although the K-G negative energy problem turns out to be a red herring, it did lead Dirac to seek a new eq'n, which turned out to describe spin ½ particles like the electron. But it, too, turned out to have negative energy sol'ns! Such is science. This time Dirac believed the eq'n & postulated the existence of the positron (a hole in the Dirac sea; see ref's.)

The KG problem w/ negative energy density arose because it's a 2nd order eq'n, so Dirac looked for a 1st **order eq'n**. When all is said and done (see ref's), the electron field $\psi(x)$ is a 4-component spinor field

$$\psi(x) = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

described by the eq'n

$$\boxed{(i\gamma^{\mu}\partial_{\mu} - m)\psi(x) = 0}$$ Dirac eq'n

where $\gamma^{\mu} = (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$

each of which is a **4x4 matrix**, in the spinor space. The mass m is understood to be multiplied by a 4x4 identity matrix. To be explicit, $i\gamma^0\partial_0$ means

$$\begin{pmatrix} \gamma^0_{11} & \gamma^0_{12} & \gamma^0_{13} & \gamma^0_{14} \\ \gamma^0_{21} & \gamma^0_{22} & \cdots & \\ \vdots & & & \\ \gamma^0_{41} & \cdots & & \gamma^0_{44} \end{pmatrix} \begin{pmatrix} i\overset{\partial/\partial t}{\partial_0}\psi_1 \\ i\partial_0\psi_2 \\ i\partial_0\psi_3 \\ i\partial_0\psi_4 \end{pmatrix}$$

Nomenclature: There are two kinds of index here:

- Lorentz index: this is the plain old index for 4-vector component as in ∂^{μ} $(\mu = 0,1,2,3)$
- Dirac index: the one corresponding to the spinor space, e.g. ψ_i $(i = 1,2,3,4)$. (I don't guarantee that I'll always use latin for these...)

so, $\underset{\substack{\uparrow \\ \text{Dirac}}}{\gamma^{\mu \leftarrow \text{Lorentz}}_{ij}}$ means the ij element of γ^{μ}.

More nomenclature: A four-vector contracted with the γ-matrices is written as

$$\gamma^\mu a_\mu \equiv \not{a} \qquad (\text{"a slash"})$$

so we can write the Dirac eq'n as

$$(i\not{\partial} - m)\psi(x) = 0.$$

(This isn't usually done when γ's are contracted with each other though.)

More on γ matrices

Notice that I haven't told you what the γ matrices are yet. That's because it doesn't matter exactly; they need only satisfy the anti-commutation relations

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

oops; I lied. They also satisfy $(\gamma^0)^2 = I$, $(\gamma^i)^2 = -I$, $i=1,2,3$ and hermitici

$\gamma^{0\dagger} = \gamma$

$\gamma^{i\dagger} = -\gamma$

$= \gamma^0\gamma^i\gamma$

Where $g^{\mu\nu}$ is as defined above and again $I_{4\times4}$ is assumed to multiply the r.h.s. This anticommutation rel'n is a consequence of requiring $E^2 = p^2 + m^2$; see any discussion of the Dirac eq'n.

There are a bunch of useful relations + trace theorems involving the γ matrices that we reproduce here from Quigg. The traces show up in matrix element calculations; All of the relations follow from the anticommutation rel'ns:

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(If they're illegible on the copies, you can find them in any field theory or gauge theory book.)

A.2 Dirac Matrices

The Dirac γ-matrices satisfy the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \tag{A.2.1}$$

so that

$$\gamma^\mu\gamma_\mu = 4 \cdot I, \tag{A.2.2}$$

where I is the 4×4 identity matrix. In contexts in which the matrix character of I is obvious, it will be convenient simply to write 1. Other useful identities follow at once from the anticommutation relations:

$$[\gamma^\mu\gamma^\nu, \gamma^\rho] \equiv \gamma^\mu\gamma^\nu\gamma^\rho - \gamma^\rho\gamma^\mu\gamma^\nu = 2(\gamma^\mu g^{\nu\rho} - \gamma^\nu g^{\mu\rho}); \tag{A.2.3}$$

$$\gamma^\mu\gamma_\nu\gamma_\mu = -2\gamma_\nu; \tag{A.2.4}$$

$$\gamma^\mu\gamma_\nu\gamma_\rho\gamma_\mu = 4g_{\nu\rho}; \tag{A.2.5}$$

$$\gamma^\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu = -2\gamma_\sigma\gamma_\rho\gamma_\nu; \tag{A.2.6}$$

$$\gamma^\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\tau\gamma_\mu = 2(\gamma_\tau\gamma_\nu\gamma_\rho\gamma_\sigma + \gamma_\sigma\gamma_\rho\gamma_\nu\gamma_\tau). \tag{A.2.7}$$

The spin tensor is

$$\sigma^{\mu\nu} = (i/2)[\gamma^\mu, \gamma^\nu] = i(\gamma^\mu\gamma^\nu - g^{\mu\nu}), \tag{A.2.10}$$

for which

$$[\sigma^{\mu\nu}, \gamma^\rho] = 2i(\gamma^\mu g^{\nu\rho} - \gamma^\nu g^{\mu\rho}). \tag{A.2.11}$$

$$\begin{aligned} \gamma^5 &\equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma_0\gamma_1\gamma_2\gamma_3 \equiv \gamma_5 \\ &= (i/4!)\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \\ &= (i/4!)\varepsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma, \end{aligned} \tag{A.2.15}$$

where the Levi-Cività tensor is defined as

$$\varepsilon_{\mu\nu\rho\sigma} = \begin{cases} +1, & \text{for even permutations of 0123} \\ -1, & \text{for odd permutations} \\ 0, & \text{otherwise,} \end{cases} \tag{A.2.16}$$

and $\varepsilon^{\mu\nu\rho\sigma} = -\varepsilon_{\mu\nu\rho\sigma}$. Evidently

$$(\gamma_5)^2 = I, \tag{A.2.17}$$

and

$$\{\gamma^5, \gamma^\mu\} = 0. \tag{A.2.18}$$

Some useful results from tensor calculus are these:

$$g^{\lambda\mu}g_{\mu\nu} = \delta^\lambda_\nu = \begin{cases} 1, & \lambda = \nu \\ 0, & \text{otherwise} \end{cases} \tag{A.3.12}$$

$$-\varepsilon^{\alpha\lambda\mu\nu}\varepsilon_{\alpha\rho\sigma\tau} = \delta^\lambda_\rho(\delta^\mu_\sigma\delta^\nu_\tau - \delta^\mu_\tau\delta^\nu_\sigma) - \delta^\lambda_\sigma(\delta^\mu_\rho\delta^\nu_\tau - \delta^\mu_\tau\delta^\nu_\rho) + \delta^\lambda_\tau(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho); \tag{A.3.13}$$

$$-\varepsilon^{\alpha\beta\mu\nu}\varepsilon_{\alpha\beta\sigma\tau} = 2(\delta^\mu_\sigma\delta^\nu_\tau - \delta^\mu_\tau\delta^\nu_\sigma); \tag{A.3.14}$$

$$-\varepsilon^{\alpha\beta\gamma\nu}\varepsilon_{\alpha\beta\gamma\tau} = 6\delta^\nu_\tau; \tag{A.3.15}$$

$$-\varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{\alpha\beta\gamma\delta} = 24. \tag{A.3.16}$$

In evaluating the traces of products of γ-matrices that occur in the computation of transition matrix elements, the following theorems are useful:

$$\mathrm{tr}(I) = 4 \longleftarrow \tag{A.3.1}$$
$$\mathrm{tr}(AB) = \mathrm{tr}(BA) \tag{A.3.2}$$
$$\mathrm{tr}(\gamma_\mu) = 0 \tag{A.3.3}$$
$$\mathrm{tr}(\text{odd number of } \gamma\text{'s}) = 0 \longleftarrow \tag{A.3.4}$$
$$\mathrm{tr}(\gamma_\mu \gamma_\nu) = 4g_{\mu\nu} \longleftarrow \tag{A.3.5a}$$
$$\mathrm{tr}(\not{a}\not{b}) = 4a \cdot b \longleftarrow \tag{A.3.5b}$$
$$\longrightarrow \mathrm{tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) = 4[g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho}] \tag{A.3.6a}$$
$$\longrightarrow \mathrm{tr}(\not{a}\not{b}\not{c}\not{d}) = 4[(a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c)] \tag{A.3.6b}$$
$$\mathrm{tr}(\gamma_5) = 0 \tag{A.3.7}$$
$$\mathrm{tr}(\gamma_5 \gamma_\mu) = 0 \tag{A.3.8}$$
$$\mathrm{tr}(\gamma_5 \gamma_\mu \gamma_\nu) = 0 \tag{A.3.9}$$
$$\mathrm{tr}(\gamma_5 \gamma_\mu \gamma_\nu \gamma_\rho) = 0 \tag{A.3.10}$$
$$\mathrm{tr}(\gamma_5 \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) = 4i\varepsilon_{\mu\nu\rho\sigma} \tag{A.3.11a}$$
$$\mathrm{tr}(\gamma_5 \not{a}\not{b}\not{c}\not{d}) = 4i\varepsilon_{\mu\nu\rho\sigma} a^\mu b^\nu c^\rho d^\sigma \tag{A.3.11b}$$

Derivations and extensions of these results may be found in many places, including Section 7.2 of Bjorken and Drell[1] and Section 28 of Berestetskii *et al.*[2]

Still more nomenclature: $\psi^\dagger$ is the hermitian conjugate of ψ, but a more useful entity is

$$\bar{\psi} \equiv \psi^\dagger \gamma^0 \qquad (\text{"}\psi\text{ bar"})$$

and since $(\gamma^0)^2 = 1$, $\psi^\dagger = \bar{\psi}\gamma^0$.

Probability current

The conserved probability current (s.t. $\partial^\mu j_\mu = 0$) is

$$j^\mu = \bar{\psi} \gamma^\mu \psi$$

and if we throw in a factor of the charge, this is the QFT version of the EM current.

Plane wave sol'ns (to Dirac eq'n)

The free Dirac eq'n $(i\not{\partial} - m)\psi(x) = 0$ has plane wave sol'ns of definite momentum p:

$$\psi(x) = (\text{const}) \begin{Bmatrix} u_r(p) \\ v_r(p) \end{Bmatrix} e^{\mp ip\cdot x}$$

where the upper choice $u_r(p)e^{-ip\cdot x}$ is for particles + the lower choice is for antiparticles.

$u_r(p)$ = 4-component spinor that doesn't depend on x but does depend on momentum p.

r = 1, 2 labels two independent sol'ns for each mom. p; it corresponds to <u>spin projection</u>. (= helicity for mass[illegible] [illegible])

similarly for $v_r(p)$

Some facts about u + v;

$$\left.\begin{array}{ll} (\not{p} - m)u_r(p) = 0 & (\not{p} + m)v_r(p) = 0 \\ \bar{u}_r(\not{p} - m) = 0 & \bar{v}_r(\not{p} + m) = 0 \end{array}\right\} \text{Follows fr/ Dirac eq'n}$$

Normalization:

$$\bar{u}_r(p)u_r(p) = 2m \qquad u_r^{+}u_s = v_r^{+}v_s = 2E\delta_{rs}\ \Big\}$$

([illegible] over r $\quad \bar{v}_r(p)v_r(p) = -2m$

<u>N.B.</u> Sometimes you see $u_r^{+}u_s = \frac{E}{m}\delta_{rs}$ in which case $\bar{u}_r u_s = \delta_{rs}$

Also $u_r^\dagger(p)\, v_s(-p) = 0$; $\bar{u}_r(p)\, v_s(p) = \bar{v}_r(p)\, u_s(p) = 0$

Useful rel'ns that follow fr/ Dirac eq'n:

$$\left(\frac{\not p + m}{2m}\right) u_r(p) = u_r(p) \qquad \left(\frac{-\not p + m}{2m}\right) v = v$$

$$\bar{u}_r(p) \left(\frac{\not p + m}{2m}\right) = \bar{u}_r(p) \qquad \bar{v}\left(\frac{-\not p + m}{2m}\right) = \bar{v}$$

Note that this means that $\pm \not p + m$ are <u>projection operators</u> onto particle + antiparticle states (modulo a normalization factor of $2m$).

<u>Ex</u> As an exercise, let's show that ⓐ $(\not p - m)\, u_r(p) = 0$ follows fr/ the Dirac eq'n; ⓑ $\bar{u}_r(p)(\not p - m) = 0$ follows from ⓐ, and ⓒ $\frac{\not p + m}{2m}$ and $\frac{-\not p + m}{2m}$ are projection operators.

ⓐ Dirac:

$$(i\not\partial - m)\, u_r(p)\, e^{-ip\cdot x} = 0 \qquad \text{(const. dropped)}$$

$$= (i\gamma^\mu \partial_\mu - m)\, u_r(p)\, e^{-ip\cdot x}$$

$$= \left[i\gamma^\mu(-ip_\mu) - m\right] u_r(p)\, e^{-ip\cdot x}$$

$$= (\not p - m)\, u_r(p)\, e^{-ip\cdot x}$$

$$= 0 \Rightarrow (\not p - m)\, u_r(p) = 0$$

$p \cdot x = p_\mu x^\mu$

$\partial_\mu(p\cdot x) = \frac{\partial}{\partial x^\mu}(p_\nu x^\nu) \overset{= p_\nu \delta^\nu_\mu}{=} p_\mu$

ex. cont.

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ⓑ Show $\bar{u}_r(p)(\not{p}-m)=0$:

We know $(\not{p}-m)u_r(p)=0$

$$\text{or} \quad (\gamma^\mu p_\mu - m)\,u_r(p)=0$$

Taking hermitian conjugate,

$$u_r^\dagger(p)\left(\gamma^{\mu\dagger} p_\mu - m\right)=0 \qquad \gamma^{\mu\dagger}=\gamma^0\gamma^\mu\gamma^0$$

$$\gamma^0\gamma^0=1$$

$$= u_r^\dagger(p)\left[\gamma^0\gamma^\mu\gamma^0 p_\mu - \gamma^0\gamma^0 m\right]$$

$$= \bar{u}_r(p)(\not{p}\gamma^0 - m\gamma^0)=0$$ & multiplying by γ^0 on r.h.s.

$$= \bar{u}_r(p)(\not{p}-m)=0$$

ⓒ Let $\Lambda_+ \equiv \frac{\not{p}+m}{2m}$ & $\Lambda_- = \frac{-\not{p}+m}{2m}$. must show

i) $\Lambda_+^2=\Lambda_+$; $\Lambda_-^2=\Lambda_-$

$$\Lambda_+^2 = \frac{(\not{p}+m)(\not{p}+m)}{(2m)(2m)} = \frac{\not{p}\not{p}+2m\not{p}+m^2}{(2m)^2}$$

$$\not{p}\not{p}=p^2=m^2$$ (note: I implied here)

$$= \frac{m^2+2m\not{p}+m^2}{(2m)^2} = \frac{2m(\not{p}+m)}{(2m)^2} = \frac{\not{p}+m}{2m}$$

similarly for Λ_-

ii) $\Lambda_+\Lambda_- = \Lambda_-\Lambda_+ = 0$

$$\Lambda_+\Lambda_- = \frac{(\not p+m)(-\not p+m)}{(2m)^2} = \frac{-\not p\not p+m^2}{2m} = \frac{-p^2+m^2}{2m} = 0$$

similarly for $\Lambda_-\Lambda_+$

iii) $\Lambda_+ + \Lambda_- = 1$ (hence the normalization)

$$\Lambda_+ + \Lambda_- = \frac{\not p+m-\not p+m}{2m} = \frac{2m}{2m} = 1$$

End of Ex.

I chose these examples because the free particle spinors u + v show up in matrix elements for processes with fermions.

One more spinor fact! important

$$\sum_{r=1}^{2} u_r(p)\bar{u}_r(p) = \not p + m$$

and $$\sum_{r=1}^{2} v_r(p)\bar{v}_r(p) = -\not p + m$$

N.B. For massless particles $\not p u = \not p v = 0$ because the m's drop out. What happens to the norm. we'll worry about later, if necessary.

Exercise: Check that I got the normalization right.

* * *

End of review of K-G + Dirac eq'ns. On to QFT (sort of)

Dirac Lagrangian and Feynman rules for QED

(1) Dirac Lagrangian + friends

Now we're going to jump into field theory, and what I say here should be seen as a recipe, not a derivation. Aitchison + Hey has a nice introduction to this stuff if you've never seen it before; otherwise look in your favorite QFT book (mine is Mandl + Shaw). Fortunately we can get away with it at this level.

Our wave functions ϕ + ψ are now to be taken as quantum fields. The quanta of these fields are the corresponding particles. A^μ is the field that goes with the photon. The fields satisfy, respectively, the K-G, Dirac, + Maxwell's eq'ns. These eq'ns of motion will be modified when we put in interactions.

So, in general we have a Lagrangian density $\mathcal{L}$, s.t. $L = \int \mathcal{L} d^3x$ and $S = \int L dt = \int \mathcal{L} d^4x$.

The Euler-Lagrange eq'ns come from requiring stationarity of the action S under variations of the fields. Let φ denote a generic field. The E-L eq'ns are

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right] = 0$$

For the free $\overset{\text{real}}{\wedge}$ Klein Gordon field,

$$\mathcal{L}_{KG} = \frac{1}{2}\left[(\partial^\mu\phi)(\partial_\mu\phi) - m^2\phi^2\right]$$

gives $$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi$$

$$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right] = \partial_\mu\left[\partial^\mu\phi\right] = \Box\phi$$

$$\Rightarrow (-\Box - m^2)\phi = 0 = (\Box + m^2)\phi$$

side calc:

$$(\partial^\mu\phi)(\partial_\mu\phi) = g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi)$$

$$\frac{\partial}{\partial(\partial_\alpha\phi)}\left[\downarrow\right] = g^{\mu\nu}\left[\delta_{\alpha\mu}\partial_\nu\phi + \partial_\mu\phi\,\delta_{\alpha\nu}\right] = 2(\partial^\alpha\phi)$$

For the Dirac field Ψ:

$$\boxed{\mathcal{L}_{Dirac} = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi}$$

Note that $\bar{\Psi}$ and Ψ count as 2 independent fields, so we get 2 eq'ns of motion. Of course, the eq'ns of motion are not independent, but are related via hermitian conjugation. The point is that we ignore $\bar{\Psi}$ in taking partial deriv's w.r.t. Ψ, and vice versa. Varying $\mathcal{L}$ w.r.t. $\bar{\Psi}$ gives Ψ eq'n

$$\frac{\partial\mathcal{L}}{\partial\bar{\Psi}} = (i\partial\!\!\!/ - m)\Psi \qquad \Rightarrow (i\partial\!\!\!/ - m)\Psi = 0$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\Psi})} = 0$$

For completeness we'll throw in the free EM Lagrangian. It's the same as the classical EM Lagrangian density (see Jackson):

$$\mathcal{L}_{\substack{EM \\ free}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

$$= -\frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

and if you're careful w/ indices you get

$$\partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = 0$$

$$\text{or} \quad \Box A^\nu - \partial^\nu (\partial_\mu A^\mu) = 0,$$

our old friend without the current. The current comes from throwing in interactions. But first,

- $\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$ is the generic form for gauge field Lagrangians. But for nonabelian theories, the F's have extra terms that vanish for U(1), i.e. for E+M. Furthermore, the F's are, in general, matrices, and to get $\mathcal{L}$, which is a (Lorentz-invariant) number, you have to take a trace.

- $\mathcal{L} = -\frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\nu A_\mu - \partial_\mu A_\nu)$ is also almost the generic form for vector (i.e. spin-1) field Lagrangians, even if they're not gauge fields. In general, you also have a mass term $+\frac{1}{2} m^2 A^\mu A_\mu$.

② EM interaction + Feynman rules for QED

Since we'll be doing electron calculations, let's just consider Dirac particles.

We can get the form of the EM interaction by requiring invariance of ψ under local $U(1)$ transformations, as we did for QM. The bottom line is the same; just substitute the covariant derivative: (cf p. 2.17)

$$\partial^\mu \to D^\mu = \partial^\mu + iqA^\mu$$

$$\Rightarrow \quad \mathcal{L} = \bar\psi(i\not\partial - q\not A - m)\psi$$

$$= \bar\psi(i\not\partial - m)\psi \quad - qA_\mu \bar\psi\gamma^\mu\psi$$

$$= \mathcal{L}_{\substack{free \\ Dirac}} - A_\mu J^\mu \qquad \text{where } J^\mu = q\bar\psi\gamma^\mu\psi$$

Adding in the term for the field itself we get

$$\boxed{\mathcal{L}_{QED} = \mathcal{L}_{\substack{free \\ Dirac}} - J^\mu A_\mu - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}}$$

and we get interaction terms in the eq'ns of motion.

What we really want are the Feynman rules, and we're not going to derive them, but, following Quigg (sec. 3.6) just give a recipe.

The Feynman rules tell you how to go from a diagram to the corresponding matrix element (or amplitude), which then goes into a cross section or width calc.

There are 3 kinds of ingredients: external lines, internal lines, + vertices. To assemble these we need to recall free particle wave fns. N.B.: It's all done in momentum space

fermion $\psi \sim u_1 e^{-ip_1 \cdot x}$

$\bar{\psi} \sim \bar{u}_2 e^{ip \cdot x}$

(we'll worry about v's later)

photon $A_\mu \sim \epsilon^*_\mu (k) e^{ik \cdot x}$

$\uparrow$

Polarization vector. There are 2 indep. polarization directions. A photon is spin 1, but because it's massless, its polarization can only have two spin projections, called transverse because $\epsilon \cdot k = 0$. We don't really care about ϵ's because we're not going to deal w/ external photons. We'll put in the rules but not the details.

On with the ingredients. They're based in part on these free wave fns.

i.) Interaction vertex : $i\mathcal{L}_{int}$

Just plug the free particle w.f.'s into $i\mathcal{L}_{int}$, taking derivatives where appropriate, then strip off the external factors (wave fns). For QED,

$$i\mathcal{L}_{int} = -iqA_\mu\bar{\Psi}\gamma^\mu\Psi$$

$$\to -iq\epsilon^*_\mu(k)\,\bar{u}_2\gamma^\mu u_1 e^{i(p_2+k-p_1)\cdot x}$$

Getting rid of external stuff leaves


$-iq\gamma^\mu$

ii.) Internal lines, or propagators

These are Green's functions for the free-field operators, e.g. $\Box+m^2$ or $i\not{\partial}-m$. The trick is that it's in momentum space, so basically you just take the inverse:

scalar: $\Box+m^2 \to p^2-m^2 \Rightarrow \text{prop} = \dfrac{i}{p^2-m^2+i\epsilon}$

where the $i\epsilon$ keeps causality happy; we can ignore it.

fermion: $\not{p}-m \to \dfrac{i}{\not{p}-m+i\epsilon} = \dfrac{i(\not{p}+m)}{p^2-m^2+i\epsilon}$


$\dfrac{i}{\not{p}-m+i\epsilon}$

For the photon we have to pick a gauge; recall

$$\Box A^{\nu} - \partial^{\nu}(\partial^{\mu} A_{\mu}) = J^{\mu}$$

Easiest is to pick Lorentz gauge: $\partial^{\mu} A_{\mu} = 0$

Then we have $\Box \to k^2$

So

$$\frac{-ig^{\mu\nu}}{k^2 + i\epsilon}$$

The $g^{\mu\nu}$ comes in because it's a vector field; the $g^{\mu\nu}$ preserves the index.

(ii) <u>External Lines</u>

These are just the free part. wave fns without the $e^{ip\cdot x}$ piece. Now we have to distinguish between u's + v's.

Fermion $\longrightarrow \overset{\text{spin}}{p, s}$ — $u_s(p)$ initial; $\bar{u}_s(p)$ final

Antifermion $\longleftarrow p, s$ — $v_s(p)$ final; $\bar{v}_s(p)$ initial

<u>NB</u>: Direction of p is to <u>right</u>; $\leftarrow$ means antiparticle

photon

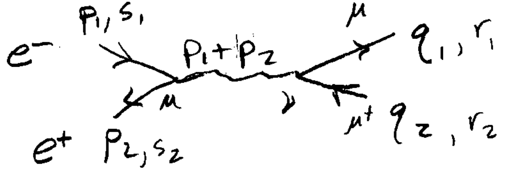
$\epsilon_{\mu}(k,\lambda)$ entering

$\epsilon^{*}_{\mu}(k,\lambda)$ leaving

Finally, in writing the factors, follow arrows <u>backwards</u>.

These aren't all the rules, but they're enough for now cross section calculated below begins p. 2.46

Example: $e^+e^- \to \mu^+\mu^-$ via single γ exchange



$p_1+p_2 = q_1+q_2$

$$\mathcal{M} = \bar{u}_{r_1}(q_1)\left(-ie\gamma^\nu\right)v_{r_2}(q_2)\,\frac{-ig_{\mu\nu}}{(p_1+p_2)^2}\,\bar{v}_{s_2}(p_2)\left(-ie\gamma^\mu\right)u_{s_1}(p_1)$$

$$= ie^2\Big[\underbrace{\bar{u}_{r_1}(q_1)\gamma_\mu v_{r_2}(q_2)}_{\text{muon piece}}\Big]\Big[\underbrace{\bar{v}_{s_2}(p_2)\gamma^\mu u_{s_1}(p_1)}_{\text{electron piece}}\Big]\Big/(p_1+p_2)^2$$

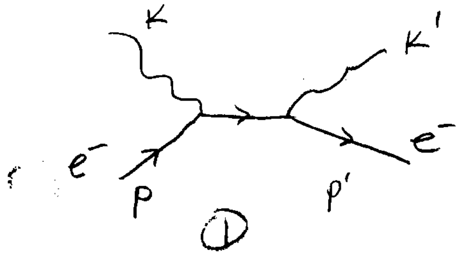
Notice that all indices, Lorentz and (implied) Dirac, are contracted, so that <u>$\mathcal{M}$ is a pure (complex) number.</u>

(In principle we can also have Z^0 exchange, see below.)

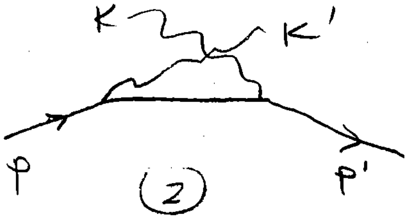
Example: Compton scattering $e\gamma \to e\gamma$

There two contributing diagrams here. Recall that we must sum the amplitudes for <u>all</u> diagrams with the same initial and final states. Initial + final states are specified by the particles, their momenta, and their spin projections. (If they have color, that counts too. A red quark is not the same as a blue quark.) } importan

So for Compton scattering there are two diagrams: one where the initial photon is absorbed and then the final photon is emitted, and one where the emission happens 1st. Viz., →



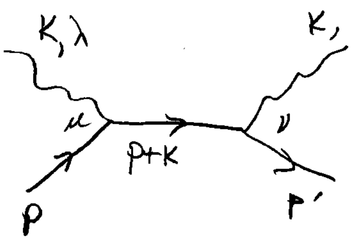
+



Note that the internal momentum is different in the two diagrams : $p+k$ in ①, and $p-k'$ in ②.

We go back to the Feynman rules to compute the amplitudes, + we get

①

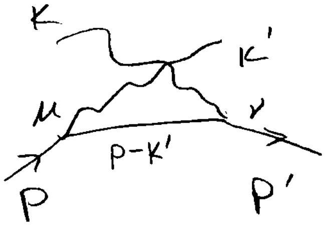


$$\mathcal{M}_1 = \bar{u}(p')(-ie\gamma^\nu)\frac{i(\not p + \not k + m_e)}{(p+k)^2 - m_e^2}(-ie\gamma^\mu)u(p)\epsilon_\mu(k) * \epsilon^*_\nu(k') \uparrow$$

$$= -ie^2\,\bar{u}(p')\not\epsilon^*(k')\frac{(\not p + \not k + m_e)}{2p\cdot k}\not\epsilon(k)\,u(p)$$

where we've been careful with placement of the $\not\epsilon$'s, remembering that this is matrix multiplication. The whole thing, though, is again simply a complex number. Similarly,

②



$$\mathcal{M}_2 = \bar{u}(p')(-ie\gamma^\nu)\frac{i(\not p - \not k' + m_e)}{(p-k')^2 - m_e^2}(-ie\gamma^\mu)\bar{u}(p)\epsilon_\nu(k) * \epsilon^*_\mu(k')$$

$$= ie^2\,\bar{u}(p')\not\epsilon(k)\frac{(\not p - \not k' + m_e)}{2p\cdot k}\not\epsilon(k')\,u(p)$$

end of Compton scatt. ex.