Cross section for $e^+e^- \to \mu^+\mu^-$

Armed with the amplitudes, we can finally break down and calculate an actual cross section.
We'll do $e^+e^- \to \mu^+\mu^-$ because it's easy (two-body final state + only 1 diagram in QED).

Recall (cf p. 1.33) the structure of the cross sect:

$$d\sigma = (\text{flux})\, |M|^2 * \text{phase space} * \delta\text{-fn}$$

we'll divide up the calculation into

1. preliminaries (kinematics etc)
2. calculation of $|M|^2$ fr/ Feynman diagrams
3. phase space integration + cross sect.

1. Preliminaries

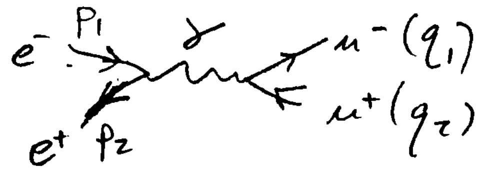
We want

$$e^-(p_1) + e^+(p_2) \to \mu^-(q_1) + \mu^+(q_2)$$

with $E_{cm} \gg m_\mu$ so we'll <u>neglect e, μ masses.</u>

We also want $E_{cm} < M_Z$, so we can <u>neglect Z exchange</u> (see below)

So the only contributing diagram is via γ exchange:



as on page 2.44. We'll <u>work in the e^+e^- center of mass frame</u>, but keep things general as much as possible, i.e., express in terms of Lorentz invariants.

<u>Kinematics</u>: Note that kinematic constraints give us useful info. Consider the final state: there are in principle 6 degrees of freedom — the components of the 3-momenta of the μ^- + μ^+ $(\vec{q}_1, \vec{q}_2)$. But conservation of 4-momentum gives 4 constraints:

$$p_1 + p_2 = q_1 + q_2$$

$\Rightarrow$ 2 independent degrees of freedom in the final state.

So to get the total cross section, we'll only have to do two integrals (the δ^4-fn will take care of the rest).

Let $s \equiv E_{cm}^2$. Then

$$s = (p_1 + p_2)^2 = (q_1 + q_2)^2$$

or $s = 2p_1 \cdot p_2 = 2q_1 \cdot q_2$

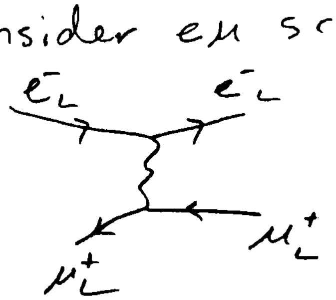
... and we've taken the plane of the momenta to define $\varphi = 0$.

N.B.: E_{cm} is considered to be fixed, so the two degrees of freedom here are the angles θ + φ

there's azimuthal symmetry, i.e., symmetry about the e^+e^- axis, so we expect no φ dependence. That makes the angular integrations easier.

Helicity argument: Not only that, but we can predict the form of the cross section with a helicity argument (+ a little hand waving; see Perkins sec. 6.6 + appendix C for more rigor).
The point is that this is a vector interaction (spin-1) and helicity is conserved (we're neglecting masses, so the e^+, e^-, μ^+ + μ^- have definite helicities).

First consider $e\mu$ scattering: $e^-\mu^+ \to e^-\mu^+$



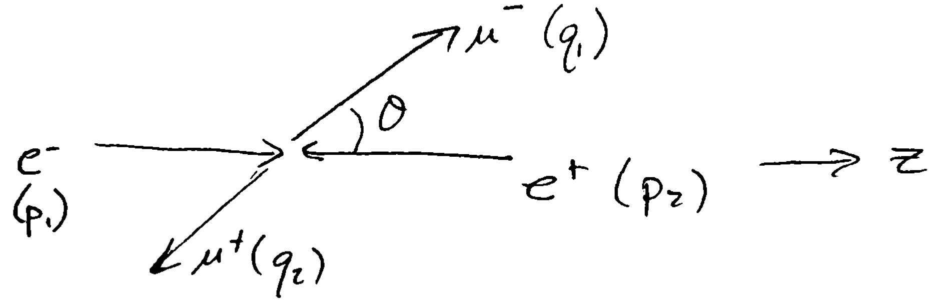
+ similarly for R

Helicity is conserved, so $e^-_L \to e^-_L$ but not e^-_R.
Similarly for the μ^+.

Now we can turn e^-_L in the final state to e^+_R in the initial state, + vice versa for the μ:

So far, this is Lorentz-inv. and true in any frame. Life is even simpler if we go to the e^+e^- CM frame. Then the e^+ + e^- collide head-on with equal + opposite momenta, and the μ^+ + μ^- go off back-to-back, also w/ equal and opposite momenta, but in some other direction. If we neglect masses, all particles have the same energy, $\frac{\sqrt{s}}{2} = \frac{E_{cm}}{2}$.

So let's neglect masses, + let θ be the angle the μ^- makes w/ e^- direction:



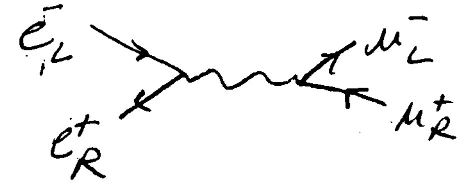
Taking the e^- to be in the $+z$ direction, we have

$$p_1 = \left(\frac{E_{cm}}{2}, 0, 0, \frac{E_{cm}}{2}\right) \quad : e^- \qquad \left(\vec{p}_1 = -\vec{p}_2\right)$$

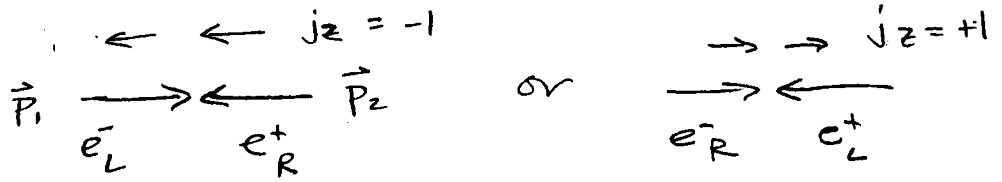
$$p_2 = \left(\frac{E_{cm}}{2}, 0, 0, -\frac{E_{cm}}{2}\right) \quad : e^+$$

$$q_1 = \left(\frac{E_{cm}}{2}, 0, \frac{E_{cm}}{2}\sin\theta, \frac{E_{cm}}{2}\cos\theta\right) \quad : \mu^- \qquad \left(\vec{q}_1 = -\vec{q}_2\right)$$

$$q_2 = \left(\frac{E_{cm}}{2}, 0, -\frac{E_{cm}}{2}\sin\theta, -\frac{E_{cm}}{2}\cos\theta\right) \quad ; \mu^+$$



The incoming $e^- + e^+$ have opposite helicities, as do the outgoing $\mu^+ + \mu^-$. So total ang. momentum has z component $J_z = -1$ or $J_z = +1$



but not $J_z = 0$. Now, as you recall fr/ PS8 1, EM interactions conserve parity, so $J_z = \pm 1$ must have equal probabilities (they can't interfere, though, any more than final states w/ different momenta can interfere - phase space integration is an incoherent sum). In the e^+e^- cm frame, the amplitude for $J_z = +1$ with the μ^- at angle θ from the e^- dir. is the "d-function" (see Appendix C of Perkins) or rotation matrix

$$d^j_{m_1 m_2}(\theta) = d^1_{+1,+1} = \tfrac{1}{2}(1+\cos\theta)$$

$J_z = 1$ we can get fr/ $\theta \to \pi - \theta \Rightarrow \tfrac{1}{2}(1-\cos\theta)$

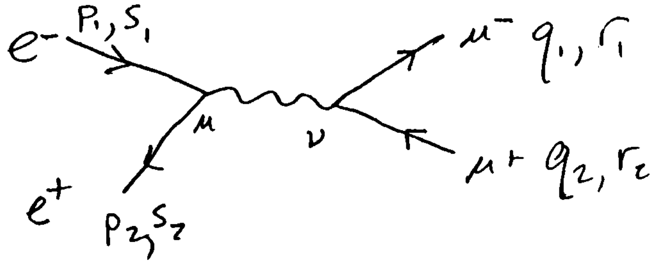
Then we square + add incoherently to get

$$\frac{d\sigma}{d\Omega} \sim (1+\cos\theta)^2 + (1-\cos\theta)^2 = 1 + \cos^2\theta$$

We'll see that this is correct.

② Calculation of $|\mathcal{M}|^2$ from Feynman diagr.

From p. 2.44,



$$s = (p_1+p_2)^2 = (q_1+q_2)^2$$

$$\mathcal{M} = \bar{u}_{r_1}(q_1)\,(-ie\gamma^\mu)\,v_{r_2}(q_2)\,\frac{-ig_{\mu\nu}}{(p_1+p_2)^2}\,\bar{v}_{s_2}(p_2)(-ie\gamma^\mu)u_{s_1}(p_1)$$

$$= \frac{ie^2\left[\bar{u}_{r_1}(q_1)\gamma_\mu v_{r_2}(q_2)\right]\left[\bar{v}_{s_2}(p_2)\gamma^\mu u_{s_1}(p_1)\right]}{(p_1+p_2)^2}$$

1st [ ]: μ's only

2nd [ ]: e's only

Now we need $\mathcal{M}^*$ for $|\mathcal{M}|^2$. Recall that

$(\gamma^\nu)^\dagger = \gamma^0\gamma^\nu\gamma^0$ so

$$\mathcal{M}^* = \frac{-ie^2\left[\overbrace{v^\dagger_{r_2}(q_2)\,\gamma^0}^{\bar{v}}\gamma_\nu\overbrace{\gamma^0\gamma^0}^{1}u_{r_1}(q_1)\right]\left[u^\dagger_{s_1}(p_1)\gamma^0\gamma^\nu\gamma^0\gamma^0 v_{s_2}(p_2)\right]}{(p_1+p_2)^2}$$

$$= \frac{-ie^2\left[\bar{v}_{r_2}(q_2)\gamma_\nu u_{r_1}(q_1)\right]\left[\bar{u}_{s_1}(p_1)\gamma^\nu v_{s_2}(p_2)\right]}{(p_1+p_2)^2}$$

Now, before we combine $\mathcal{M}$ + $\mathcal{M}^*$, we have to say something about helicity sums. If everything is **unpolarized**, we

- <u>average</u> over initial helicities $\left(\frac{1}{2}\sum_{s_1}\right)\left(\frac{1}{2}\sum_{s_2}\right)$

- <u>sum</u> over final helicities $\sum_{r_1}\sum_{r_2}$

That is, we don't use polarized initial state, + we don't measure the μ pol. in the final state (i.e., accept all polarizations). Note that these are incoherent sums; different helicity states don't interfere.

These sums over helicity states will allow us to use the identities discussed above for the Dirac eq'n.

So we want

$$\overline{|\mathcal{M}|^2} \equiv \frac{1}{4}\sum_{\substack{s_1 s_2 \\ r_1 r_2}} \mathcal{M}^* \mathcal{M} = \frac{1}{4}\frac{e^4}{(p_1+p_2)^4} \underbrace{\sum_{r_1 r_2}\left[\bar{v}_{r_2}(q_2)\gamma_\nu u_{r_1}(q_1)\right]\left[\bar{u}_{r_1}(q_1)\gamma_\mu v_{r_2}(q_2)\right]}_{\equiv A_{\nu\mu}}$$

$$\underbrace{\sum_{s_1 s_2}\left[\bar{u}_{s_1}(p_1)\gamma^\nu v_{s_2}(p_2)\right]\left[\bar{v}_{s_2}(p_2)\gamma^\mu u_{s_1}(p_1)\right]}_{\equiv B^{\nu\mu}}$$

Note that I've arranged the μ + e stuff s.t. $A_{\nu\mu}$ has only muon momenta, + $B^{\nu\mu}$ has only e momenta.

Stated another way, $A_{\nu\mu}$ has to do with the final state (or the $\mu^+\mu^-$ vertex) and $B^{\nu\mu}$ has to do with the initial state (or the e^+e^- vertex). We can calculate them separately, then contract them to get $\overline{|M|^2}$.

So we'll do each in turn. It will turn out that $A_{\nu\mu}$ (+ $B^{\nu\mu}$) is a trace in Dirac space. First we have to use our identities, though. It'll help to write the Dirac indices explicitly to see what happens. Note that with the indices explicit, each element is just a number, and we can rearrange to our hearts' content. Let's use $a, b, c, \ldots$ to refer to Dirac indices (remember they go from 1 to 4 and repeated indices are summed)

$$A_{\nu\mu} = \sum_{r_1}\sum_{r_2} \bar{v}_{r_2 a}(q_2)(\gamma_\nu)_{ab}\, u_{r_1 b}(q_1)\, \bar{u}_{r_1 c}(q_1)(\gamma_\mu)_{cd}\, v_{r_2 d}(q_2)$$

$$= \underbrace{\left[\sum_{r_2} v_{r_2 d}(q_2)\bar{v}_{r_2 a}(q_2)\right]}_{=-(\not{q}_2)_{da}}(\gamma_\nu)_{ab}\underbrace{\left[\sum_{r_1} u_{r_1 b}(q_1)\bar{u}_{r_1 c}(q_1)\right]}_{=(\not{q}_1)_{bc}}(\gamma_\mu)_{cd} \quad \Big\}\, m=0$$

$$= -\not{q}_{2\,da}\,\gamma_{\nu\,ab}\,\not{q}_{1\,bc}\,\gamma_{\mu\,cd}$$

$$\equiv \mathrm{Tr}(\not{q}_2\gamma_\nu\not{q}_1\gamma_\mu)$$

check signs...

As promised, $A_{\nu\mu}$ is just a trace. We can write it as

$$A_{\nu\mu} = q_2^{\alpha} q_1^{\beta} \, Tr(\gamma_\alpha \gamma_\nu \gamma_\beta \gamma_\mu) \qquad \text{\& use our trace thms}$$

$$= -4 q_2^{\alpha} q_1^{\beta} \left[g_{\alpha\nu} g_{\beta\mu} - g_{\alpha\beta} g_{\nu\mu} + g_{\alpha\mu} g_{\nu\beta} \right]$$

$$= -4 \left[q_{2\nu} q_{1\mu} - q_1 \cdot q_2 \, g_{\nu\mu} + q_{2\mu} q_{1\nu} \right]$$

Note that $A_{\nu\mu}$ is symmetric in μ & ν. That is, $A_{\nu\mu} = A_{\mu\nu}$. It's a good thing, because I could just as easily have switched the order of the square brackets.

$B^{\nu\mu}$ is also a trace:

$$B^{\nu\mu} = -Tr(\not{p}_1 \gamma^\nu \not{p}_2 \gamma^\mu)$$

$$= -4\left[p_1^\nu p_2^\mu - p_1 \cdot p_2 \, g^{\nu\mu} + p_1^\mu p_2^\nu \right]$$

Note that $B^{\nu\mu}$ looks a lot like $A_{\mu\nu}$. That's because the interaction vertex looks the same for both.

So putting them together

$$|\mathcal{M}|^2 = \frac{e^2}{4(p_1+p_2)^4} A_{\nu\mu} B^{\nu\mu}$$

and

$$A_{\nu\mu}B^{\nu\mu} \overset{16}{=} (q_{2\nu}q_{1\mu} + q_{2\mu}q_{1\mu} - q_1\cdot q_2\, g_{\nu\mu})(p_2^{\nu}p_1^{\mu} + p_2^{\mu}p_1^{\nu} - p_1\cdot p_2\, g^{\nu\mu})$$

$$= \Big[(1+1)q_2\cdot p_2\, q_1\cdot p_1 + (1+1)q_2\cdot p_1\, q_1\cdot p_2 - (1+1)\cancel{p_1\cdot p_2\, q_1\cdot q_2}$$
$$- (1+1)\cancel{q_1\cdot q_2\, p_1\cdot p_2} + \cancel{q_1\cdot q_2\, p_1\cdot p_2\, \underbrace{g_{\nu\mu}g^{\nu\mu}}_{=4}}\Big] * 16$$

$$= 2(q_2\cdot p_2\, q_1\cdot p_1 + q_2\cdot p_1\, q_1\cdot p_2) * 16$$

So, with $(p_1+p_2)^2 = s$,

$$\boxed{|\overline{\mathcal{M}}|^2 = \frac{8e^4}{s^2}\Big[(p_1\cdot q_1)(p_2\cdot q_2) + (p_1\cdot q_2)(p_2\cdot q_1)\Big]}$$

Note that this is a real number, and it's Lorentz invariant, as is always true for $|\overline{\mathcal{M}}|^2$. It's still in general form.

<u>In the e^+e^- cm frame</u> (cf p. 2.48),

Let $E = \frac{\sqrt{s}}{2} = \frac{E_{cm}}{2}$ be the energy of any of the particles (remember they're all the same in the cm frame if we neglect masses).

with $\vec{p}_1 = -\vec{p}_2$, $\vec{q}_1 = -\vec{q}_2$, and $\vec{p}_1\cdot\vec{q}_1 = E^2\cos\theta$,

$$(p_1\cdot q_1)(p_2\cdot q_2) = E^4(1-\cos\theta)^2$$
$$(p_1\cdot q_2)(p_2\cdot q_1) = E^4(1+\cos\theta)^2$$

$\rightarrow$

So, finally, with $e^4 = \frac{s^2}{16}$,

$$\boxed{|\overline{\mathcal{M}}|^2_{CM} = e^4(1+\cos^2\theta)}$$

with angular dependence as advertised.

③ Phase space integration + cross section

Now it's time to pull it all together, + now we have to get the overall factors right (darn...). What I'm about to say is very general. I'll show what the general form for a c.s. is, what it is for any $2 \to 2$ process in the CM frame, then finally I'll plug in the $|\mathcal{M}|^2$ at the top of this page.

The differential cross section for 2 initial particles (momenta p_i) to scatter into N final particles (momenta p_f') is given by

$$d\sigma = \underbrace{\frac{1}{4E_1E_2|\vec{v}_{rel}|}}_{\text{initial flux}} |\overline{\mathcal{M}}|^2 \underbrace{\left(\prod_f \frac{d^3p_f'}{(2\pi)^3 2E_f'}\right)(2\pi)^4 \delta^4(\Sigma p_f' - \Sigma p_i)}_{dLips}$$

Lorentz invariant phase space

Where these things come from is discussed in Tipton's notes, or Mandl + Shaw p. 138-139, or Aitchison + Hey p. 142-144, or most field theory books. Note that there may be additional factors of $2m$ for fermions (e.g. in Mandl + Shaw) due to differences in normalization of spinors. I may throw in the Mandl + Shaw discussion (Xerox or my "translation," depending on my inclination) as a handout.

Note that $d\sigma$ is Lorentz invariant:

flux: $\vec{v}_{rel} = \frac{\vec{P}_2}{E_2} - \frac{\vec{P}_1}{E_1}$, + with a little algebra, we can show

$$\boxed{\left(E_2 E_1 v_{rel}\right)^2 = (P_1 \cdot P_2)^2 - m_1^2 m_2^2}$$

$\Rightarrow$ flux is Lorentz inv.

$|\mathcal{M}|^2$: Lorentz inv. in our example, and in general (cf field theory books — where $\mathcal{M}$ comes from)

dLips: Already shown a couple of lectures ago.

So far, so general. Let's go to $2 \to 2$ processes

2-body final state

Let p_1, p_2 = initial momenta

p_1', p_2' = final momenta

No assumptions about masses or frame yet. $d\sigma$ becomes

$$d\sigma_{2\to 2} = \frac{1}{64\pi^2 v_{rel} E_1 E_2 E_1' E_2'} |\overline{\mathcal{M}}|^2 d^3p_1' \, d^3p_2' \, \delta^4(p_1'+p_2'-p_1-p_2)$$

3 of the 4 δ-fns go away when we integrate over d^3p_2, + with $d^3p_1' = |\vec{p}_1'|^2 d|\vec{p}_1'| \, d\Omega_1'$ we have

$$d\sigma = \frac{1}{64\pi^2 v_{rel} E_1 E_2 E_1' E_2'} |\overline{\mathcal{M}}|^2 \, |\vec{p}_1'|^2 d|\vec{p}_1|' \, d\Omega_1' \, \delta(E_1+E_2-E_1'-E_2')$$

Where E_2' is shorthand for $E_2' = \sqrt{|\vec{p}_2'|^2 + m_2'^2}$, and $\vec{p}_2' = \vec{p}_1 + \vec{p}_2 - \vec{p}_1'$, since we've already done the $\vec{p}_2'$ integration. Note that E_2' depends on $\vec{p}_1'$, so we have to take that into account when we do the p_1 integration.

Well, $|\vec{p}_1|'$ is fixed by the energy δ-fn. Recall that for δ-fns (see, e.g., Jackson, chap. 1)

$$\int f(x)\, \delta[g(x)]\, dx = \left. \frac{f(x)}{|(\partial g/\partial x)|} \right|_{g(x)=0}$$

Here, $x = |\vec{p}_1'|$, $g(x) = E_1 + E_2 - E_1' - E_2'$, + $f(x)$ = the rest of the integrand

So we need $\left| \dfrac{\partial}{\partial |\vec{p}_1'|} (E_1 + E_2 - E_1' - E_2') \right| = \dfrac{\partial (E_1' + E_2')}{\partial |\vec{p}_1'|}$ since $E_1 + E_2$ fixed.

Now, since $|\vec{p}|^2 + m^2 = E^2$ in general,

$$2 |\vec{p}| \, d|\vec{p}| = 2 E \, dE$$

or $$\frac{\partial E}{\partial |\vec{p}|} = \frac{|\vec{p}|}{E}$$

So, $$\frac{\partial (E_1' + E_2')}{\partial |\vec{p}_1'|} = \frac{|\vec{p}_1'|}{E_1'} + \frac{|\vec{p}_2'|}{E_2'}$$

<u>Now we go</u> to the cm frame ; $|\vec{p}_1'| = |\vec{p}_2'|$

by energy cons.

$$\Rightarrow \frac{\partial (E_1' + E_2')}{\partial |\vec{p}_1'|} = \frac{|\vec{p}_1'| (E_1' + E_2')}{E_1' E_2'} = |\vec{p}_1'| \frac{(E_1 + E_2)}{E_1' E_2'}$$

Plugging back into $d\sigma$ on p. 2.58, (+ doing the $|\vec{p}_1'|$ integration we get

$$d\sigma = \frac{1}{64\pi^2 v_{rel} E_1 E_2 E_1' E_2'} \, |\mathcal{M}|^2 |\vec{p}_1'|^2 d\Omega_1' \, \frac{E_1' E_2'}{|\vec{p}_1'| (E_1 + E_2)}$$

Now, $$(v_{rel} E_1 E_2)_{cm} = \left(\frac{|\vec{p}_1|}{E_1} + \frac{|\vec{p}_1|}{E_2} \right) E_1 E_2 = |\vec{p}_1| (E_1 + E_2)$$

So finally,

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{\substack{2\to2\\ cm}} = \frac{1}{64\pi^2(E_1+E_2)^2}\frac{|\vec{p}_1'|}{|\vec{p}_1|}\,|\overline{m}|^2}$$

and, though we've gone to the cm frame, we've assumed nothing about masses, so this holds for any mass.

<u>Now, back to $e^+e^- \to \mu^+\mu^-$</u>

Recall $|\overline{m}|^2_{cm} = e^4(1+\cos^2\theta)$ (cf. p. 2.56)

With $(E_1+E_2)^2 = s = (p_1+p_2)^2$, and $\alpha = \frac{e^2}{4\pi}$

Now take <u>massless</u> case: $|\vec{p}_1'| = |\vec{p}_1|$

We have

$$\frac{d\sigma(e^+e^-\to\mu^+\mu^-)}{d\Omega_{cm}} = \frac{\alpha^2}{4s}(1+\cos^2\theta)$$

with $\int d\varphi \to 2\pi$ and $\int_{-1}^{1}\cos^2\theta\, d(\cos\theta) = \frac{2}{3}$,

$$\sigma_{tot} = \int\left(\frac{d\sigma}{d\Omega}\right)d\Omega = \frac{4}{3}\pi\frac{\alpha^2}{s}$$