# Complex Analysis Review

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# 1. Topic Review

**1.1. Holomorphic Functions.** A complex function $f(z) = u(x,y) + iv(x,y)$ is said to be Holomorphic in a region if it satisfy the Cauchy Riemann Equations

$$\begin{aligned} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \end{aligned} \tag{1}$$

in that region. Note: Analyitic and Holomorphic seem to be used interchangeably.

**1.2. Cauchy's Integral Theorm.** The Cauchy integral theorem states that if some complex function $f(z)$ is holomorphic (no poles of branch points) on some dome domain $\Omega$ in $\mathbb{C}$ then

$$\oint_{\partial\Omega} f(z)dz = 0 \tag{2}$$

**1.3. Cauchy's Integral Formula.** The Cauchy integral formula says that if some complex function $f(z)$ is holomorphic on some domain domain $\Omega$ in $\mathbb{C}$ then the function at any point $c \in \Omega$ is equal to

$$f(c) = \oint_{\partial\Omega} \frac{f(z)}{z-c} dz \tag{3}$$

**1.4. Real Series Expansions.** Obviously the taylor expansion of a real function f(x) about a is

$$f(x) = f(a) + (x-a)f'(x)|_a + (x-a)\frac{f''(x)|_a}{2!} + ... \tag{4}$$

Two key expansions

- Geometric Series

$$S_n = \sum_{n=0}^{N} x^n = \frac{1-x^{N+1}}{1-x} \quad \forall |x| < 1 \tag{5}$$

To derive this simple subtract $xS_N$ from $S_N$, and all terms will cancel expect the first and last term,

$$S_N = \sum_{n=0}^{N} x^n$$

$$xS_N = \sum_{n=0}^{N} x^{n+1}$$

$$S_N - xS_N = \sum_{n=0}^{N} x^n - x^{n+1}$$

$$(1-x)S_N = 1 - x^N$$

$$S_N = \frac{1-x^{N+1}}{1-x}$$

1.5. **Laurent Series.** The Laurent series of some function holomorphic function $f(z)$ about some location $c \in \Omega$ is defined as

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-c)^k \tag{6}$$

where

$$a_k = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{f(z)}{(z-c)^{k+1}} dz \tag{7}$$

To get some intuition as to where the coefficient equation comes from, plug in the Laurent series expansion for $f(z)$ into the above equation, and you will see that most of the coefficients will no longer have a pole because the $\frac{1}{(z-c)^{k+1}}$, and so by the Cauchy integral Theorem, these terms will be zero, and so the only contribution will come from the $a_k$ coefficient. This is not a proof just some intuition.

Now note that if there are poles or branch points, your Laurent series must be split up into the different regions so that the function $f(z)$ is holomorphic in those regions. See examples.

1.6. **Residues.** Suppose we have a complex function that has a Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-c)^k \tag{8}$$

then

$$\begin{aligned}\oint_{\partial\Omega} f(z)dz &= \oint_{\partial\Omega} \sum_{k=-\infty}^{\infty} a_k (z-c)^k \\ &= \sum_{k=-\infty}^{\infty} a_k \oint_{\partial\Omega} (z-c)^k \\ &= \sum_{k=-\infty}^{-2} a_k \oint_{\partial\Omega} (z-c)^k + a_{-1} \oint_{\partial\Omega} \frac{1}{(z-c)} + \sum_{k=0}^{\infty} a_k \oint_{\partial\Omega} (z-c)^k\end{aligned}$$

the first and last sum must be zero by Cauchy integral theorem (they contain no poles) and so

$$\oint_{\partial\Omega} f(z)dz = a_{-1} \oint_{\partial\Omega} \frac{1}{(z-c)}$$

We call $a_{-1}$ the Residue. The closed contour can be evaluated on a unit circle contour centered at c, which will produce a $2\pi i$, this leads to the formula

$$\oint_{\partial\Omega} f(z)dz = 2\pi i \quad Res\{f(z)\} \tag{9}$$

If there are multiple poles, than this terns into the sum

$$\oint_{\partial\Omega} f(z)dz = 2\pi i \sum_{z_0} Res\{f(z), z_0\} \tag{10}$$

To actually evaluate the residue of an $m^{th}$ order pole one multiplies the Laurent series by $(z-z_0)^m$, then one differentiates that m-1 times and you will "kill" every term in the Laurent series expect for the $a_{-1}$

$$a_{-1} = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z)\right]_{z_0} \tag{11}$$

or for a first order simple pole where $f(z) = \frac{g(z)}{h(z)}$

$$a_{-1} = \left[\frac{g(z)}{h'(z)}\right]_{z_0} \tag{12}$$

## 2. Contour Integrals

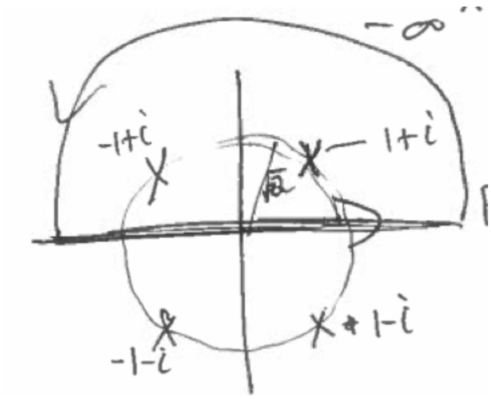
### 2.1. Physics 401 Midterm 2014.

$$\int_{-\infty}^{\infty} \frac{x^2 + ax}{x^4 + 4} dx$$

- Predictions
  1) The result should be real
  2) I cannot say if it will be positive or negative because of the linear x term (I suspect it will be positive because the leading order is $x^2$)
  3) Dimensional analysis cannot really occur here

- Poles To find poles

$$\begin{aligned} x^4 + 4 &= 0 \\ &= -4 \\ &= (4)(-1) \\ &= 4e^{i\pi + 2\pi i k} \quad \text{for k} \in [0, 1, 2, 3] \\ &= 4e^{i\pi(1+2k)} \\ x &= 4^{\frac{1}{4}} e^{\frac{i\pi}{4}(1+2k)} \\ &= \sqrt{2} e^{\frac{i\pi}{4}(1+2k)} \\ x &= \left\{ \sqrt{2}e^{\frac{i\pi}{4}}, \sqrt{2}e^{\frac{3i\pi}{4}}, \sqrt{2}e^{\frac{5i\pi}{4}}, \sqrt{2}e^{\frac{7i\pi}{4}} \right\} \end{aligned}$$




- Contour Now propose the contour in the above figure, and we hope that this contour will reduce to our real contour. Recall

$$\begin{aligned} z &= re^{i\theta} \\ dz &= e^{i\theta} dr \qquad (13) \\ dz &= ire^{i\theta} d\theta \end{aligned}$$

$$\oint \frac{z^2+az}{z^4+4}dz = \lim_{R\to\infty,\theta=0}\int_{-R}^{R} \frac{(re^{i\theta})^2+a(re^{i\theta})}{(re^{i\theta})^4+4}e^{i\theta}dr + \lim_{R\to\infty,}\int_0^{\pi} \frac{(Re^{i\theta})^2+a(Re^{i\theta})}{(Re^{i\theta})^4+4}iRe^{i\theta}d\theta$$

$$= \int_{-\infty}^{\infty} \frac{r^2+ar}{r^4+4}dr + \lim_{R\to\infty}\int_0^{\pi} \frac{R^3e^{3i\theta}+aR^2e^{2i\theta}}{R^4e^{4i\theta}+4}e^{i\theta}id\theta \to 0$$

$$\oint f(z)dz = \int_{-\infty}^{\infty} \frac{r^2+ar}{r^4+4}dr$$

where we have let the arc integral go to zero because the limit of $R\to\infty$ of $\frac{R^3}{R^4} = 0$. Now we have proven our contour integral equivalent to our original integral, hence we can say

$$\int_{-\infty}^{\infty} \frac{x^2+ax}{x^4+4}dx = \oint \frac{z^2+az}{z^4+4}dz$$

$$= 2\pi i Res\left\{\frac{z^2+az}{z^4+4}; \sqrt{2}e^{\frac{i\pi}{4}}, \sqrt{2}e^{\frac{3i\pi}{4}}\right\}$$

$$= 2\pi i \sum_{\left\{\sqrt{2}e^{\frac{i\pi}{4}}, \sqrt{2}e^{\frac{3i\pi}{4}}\right\}} \frac{z^2+az}{4z^3}$$

$$= 2\pi i\left[\frac{(\sqrt{2}e^{\frac{i\pi}{4}})^2+a(\sqrt{2}e^{\frac{i\pi}{4}})}{4(\sqrt{2}e^{\frac{i\pi}{4}})^3} + \frac{(\sqrt{2}e^{\frac{3i\pi}{4}})^2+a(\sqrt{2}e^{\frac{3i\pi}{4}})}{4(\sqrt{2}e^{\frac{3i\pi}{4}})^3}\right] \tag{14}$$

$$= 2\pi i\left[\frac{2e^{\frac{2i\pi}{4}}+a(\sqrt{2}e^{\frac{i\pi}{4}})}{4(2)\sqrt{2}e^{\frac{3i\pi}{4}}} + \frac{2e^{\frac{6i\pi}{4}}+a(\sqrt{2}e^{\frac{3i\pi}{4}})}{4(2)\sqrt{2}e^{\frac{9i\pi}{4}}}\right]$$

$$= \frac{\pi i}{4\sqrt{2}}\left[\frac{2e^{\frac{i\pi}{2}}+a(\sqrt{2}e^{\frac{i\pi}{4}})}{e^{\frac{3i\pi}{4}}} + \frac{2e^{\frac{3i\pi}{2}}+a(\sqrt{2}e^{\frac{3i\pi}{4}})}{e^{\frac{9i\pi}{4}}}\right]$$

..... Algebra

$$\int_{-\infty}^{\infty} \frac{x^2+ax}{x^4+4}dx = \frac{\pi}{2}$$

Note with the example above you life gets much easier if you recognize that our poles are just at $\sqrt{2}(1+i)$ and $\sqrt{2}(-1+i)$. When first doing this I didn't recall what sin and cos of $\frac{\pi}{4}$ was.

2.2. **Physics 401 Midterm 2014.**

$$\int_0^{\infty} \frac{x^{\frac{1}{4}}}{x^2+a^4}dx$$

– Predictions
  1) The result should be real
  2) Should be positive

3) x must have units of $a^2$ because we have an $x^2 + a^4$. So if this is the case we will have

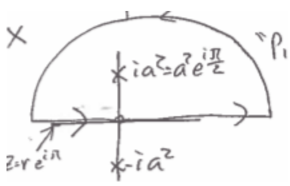
$$\frac{(a^2)^{\frac{1}{4}}}{a^4} a^2 = \frac{a^{\frac{1}{2}}}{a^2} = a^{\frac{1}{2}} a^{-\frac{4}{2}} = a^{-\frac{3}{2}}$$

– Poles To find poles

$$\begin{aligned} x^2 + a^4 &= 0 \\ x^2 &= (-1)a^4 \\ &= a^4 e^{i\pi + 2\pi k} \quad \text{for k} \in [0,1] \\ &= a^4 e^{i\pi(1+2k)} \\ x &= a^2 e^{\frac{i\pi}{2}(1+2k)} \\ x &= \left\{ a^2 e^{\frac{i\pi}{2}}, a^2 e^{\frac{3i\pi}{2}} \right\} \\ x &= \left\{ ia^2, -ia^2 \right\} \end{aligned}$$

We must also take care and recognize that we have a branch point at x=0.

– Contour Our proposed contour should look as below




We choose this because first we must get around the branch point with contour warping, but still include our real domain. This would suggest we could use the packman count our. This will work, but for this example, the half packman will work just swell.

So have

$$\oint \frac{z^{\frac{1}{4}}}{z^2 + a^4} dz = \lim_{\substack{\epsilon \to 0 \\ R \to \infty}} \left[ \int_\epsilon^R f(z) e^{i\theta} dr \right]_{\theta=0} + \left[ \cancelto{0}{\lim_{R \to \infty} \int_0^\pi f(z) iRe^{i\theta} d\theta} \right]_{r=R} + ...$$

$$\lim_{\substack{\epsilon \to 0 \\ R \to \infty}} \left[ \int_R^\epsilon f(z) e^{i\theta} dr \right]_{\theta=\pi} + \left[ \cancelto{0}{\lim_{\epsilon \to 0} \int_\pi^0 f(z) i\epsilon d\theta} \right]_{r=\epsilon}$$

The theta integrals tend to zero in the limits, this is obvious for the $R \to \infty$ line, but for the $\epsilon \to 0$ remember that the denominator is approximately just $a^4$ for small $\epsilon$ and so the limit gives zero.

$$\begin{aligned}
\oint \frac{z^{\frac{1}{4}}}{z^2+a^4}dz &= \lim_{\substack{\epsilon\to 0\\ R\to\infty}} \left[\int_\epsilon^R f(z)e^{i\theta}dr\right]_{\theta=0} - \lim_{\substack{\epsilon\to 0\\ R\to\infty}} \left[\int_\epsilon^R f(z)e^{i\theta}dr\right]_{\theta=\pi} \\
&= \lim_{\substack{\epsilon\to 0\\ R\to\infty}} \left[\int_\epsilon^R \frac{(re^{i\theta})^{\frac{1}{4}}}{(re^{i\theta})^2+a^4}e^{i\theta}dr\right]_{\theta=0} \dots. \\
&- \lim_{\substack{\epsilon\to 0\\ R\to\infty}} \left[\int_\epsilon^R \frac{(re^{i\theta})^{\frac{1}{4}}}{(re^{i\theta})^2+a^4}e^{i\theta}dr\right]_{\theta=\pi} \\
&= \int_0^\infty \frac{r^{\frac{1}{4}}}{r^2+a^4}dr - \int_0^\infty \frac{r^{\frac{1}{4}}e^{i\pi/4}}{r^2\cancelto{1}{e^{2\pi i}}+a^4}\cancelto{-1}{e^{i\pi}}dr \\
&= (1+e^{i\pi/4})\int_0^\infty \frac{r^{\frac{1}{4}}}{r^2+a^4}dr
\end{aligned}$$

And so we can conclude that

$$\begin{aligned}
\int_0^\infty \frac{x^{\frac{1}{4}}}{x^2+a^4}dx &= \frac{1}{1+e^{i\pi/4}}\oint \frac{z^{\frac{1}{4}}}{z^2+a^4}dz \\
&= \frac{2\pi i}{1+e^{i\pi/4}} Res\left\{\frac{z^{\frac{1}{4}}}{z^2+a^4}; ia^2\right\} \\
&= \frac{2\pi i}{1+e^{i\pi/4}} \sum_{ia^2} \frac{z^{\frac{1}{4}}}{2z} \\
&= \frac{2\pi i}{1+e^{i\pi/4}} \left[\frac{(ia^2)^{\frac{1}{4}}}{2(ia^2)}\right] \\
&= \frac{2\pi i}{1+e^{i\pi/4}} \left[\frac{\sqrt{a}(i)^{\frac{1}{4}}}{2(ia^2)}\right] \\
&= \frac{\cancel{2}\pi\cancel{i}}{1+e^{i\pi/4}} \frac{\sqrt{a}}{\cancel{2}\cancel{i}a^2}\left[(i)^{\frac{1}{4}}\right] \\
&= \frac{\pi a^{-\frac{3}{2}}}{1+e^{i\pi/4}} \left[(e^{\frac{i\pi}{2}})^{\frac{1}{4}}\right] \\
&= \frac{\pi a^{-\frac{3}{2}}}{1+e^{i\pi/4}} e^{\frac{i\pi}{8}} \\
&= \frac{\pi a^{-\frac{3}{2}}}{e^{\frac{i\pi}{8}}+e^{\frac{-i\pi}{8}}} \\
\int_0^\infty \frac{x^{\frac{1}{4}}}{x^2+a^4}dx &= \frac{2\pi a^{\frac{-3}{2}}}{cos(\pi/8)}
\end{aligned}$$

## 3. Laurent Series

3.1. **Physics 401 Notes. Find the series expansion of the function below about $z_0 = 0$.**

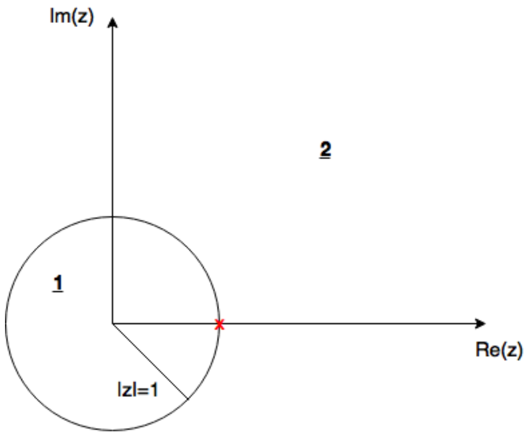
$$f(z) = \frac{1}{z-1}$$

Our proposed Laurent Series will look like

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-c)^k$$

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k$$

We begin by finding all singularities and branch points of our function, these will determine the regions of convergence. Here we see there is a simple pole at z=1. So our picture will look as below.




Clearly we now will have two regions, one where $|z| < 1$ and another where $|z| > 1$.

**Region 1**. In this region we have no branch points and no poles, so a Taylor series will suffice. One may either do out the taylor expansion or recognize that this is the geometric series (because $|z| < 1$)

$$f(z) = \frac{1}{z-1} = -\frac{1}{1-z} = -\sum_{n=0}^{\infty} z^n$$

**Region 2** Now we encounter a problem because we have a simple pole located at z=1, and so we will need the full on Laurent series. This is a trick that come up over and over again in these problems, try to rewrite the function as a geometric series in a clever way. For example in this region we are guaranteed that $|z| > 1$ and so we know that $\frac{1}{z} < 1$ will always be true, and so

$$\begin{aligned} f(z) &= \frac{1}{z-1} \\ &= \frac{1}{z(1-\frac{1}{z})} \\ &= \frac{1}{z}\ \frac{1}{1-\frac{1}{z}} \\ &= \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} \\ f(z) &= \sum_{n=0}^{\infty} z^{n+1} \end{aligned}$$

Note that this trick comes up all over the place because actually calculating the coefficients is not practical by the Cauchy integral formula.

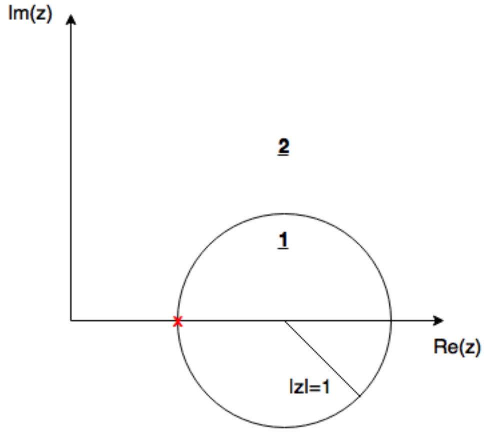
3.2. **Physics 401 Notes. Find the series expansion of the function below about $z_0 = 2$.**

$$f(z) = \frac{1}{(z-2)+1}$$

Our proposed Laurent series will look like

$$\begin{aligned} f(z) &= \sum_{k=-\infty}^{\infty} a_k(z-c)^k \\ f(z) &= \sum_{k=-\infty}^{\infty} a_k(z-2)^k \end{aligned}$$

We now identify that there is a singularity when $z = 1$. So our picture will look as below




again we expect to have a Taylor series in region 1 because there are no singularities, and a Laurent series in region 2.

**Region 1** In this region $|z-2| < 1$ so we can see that $f(z)$ could be written as a geometric series

$$\begin{aligned} f(z) &= \frac{1}{(z-2)+1} \\ &= \frac{1}{1+(z-2)} \\ f(z) &= \sum_{n=0}^{\infty} (-1)^n (z-2)^n \text{WHY?} \end{aligned}$$

**Region 2**