ICTP

www.ictp.it

Behavior of the solutions near the regular singular point.

If $\gamma_1-\gamma_2$ is not an integer, then we have the solutions

$$\phi_{\frac{1}{2}}(x)=\sum_{n=0}^{\infty} a_n^{\{\frac{1}{2}\}}(x-x_0)^{n+\gamma_{\frac{1}{2}}},$$

with
$$a_n^{\{\frac{1}{2}\}}=\frac{-\sum_{j=0}^{n-1}\left[P_{n-j}(j+\gamma_{\frac{1}{2}})+Q_{n-j}\right]a_j^{\{\frac{1}{2}\}}}{n\left(n\pm(\gamma_1-\gamma_2)\right)},$$

$$\gamma_{\frac{1}{2}}=\frac{1-P_0}{2}\pm\frac{1}{2}\sqrt{(1-P_0)^2-4Q_0}\,.$$

For x near x_0, $|x-x_0|$ is small so the $n=0$ term dominates, and

$$\phi_1(x)\approx(x-x_0)^{\gamma_1}\,,\quad \phi_2(x)\approx(x-x_0)^{\gamma_2}.$$

That is, the indices γ_1 & γ_2 give the behavior of the solutions near x_0.

If, on the other hand, $\gamma_1-\gamma_2=N$ (an integer), then still $\phi_1(x)\approx(x-x_0)^{\gamma_1}$, but the behavior of ϕ_2 requires a bit more analysis.

Recall that we can find ϕ_2 as

$$\phi_2(x) = \phi_1(x) \int^x \frac{w(x')}{\phi_1^2(x')} dx', \quad \text{with } w(x) = e^{-\mathcal{P}(x)}, \quad \mathcal{P}(x) = \int^x p(x')dx'.$$

Let us apply this prescription in the neighborhood of x_0 to get the behavior of ϕ_2 there.

$$\mathcal{P}(x) = \int^x p(x')dx' = \sum_{i=0}^{\infty} P_i \int (x'-x_0)^{i-1} dx'$$

$$\approx P_0 \int^x \frac{dx'}{x'-x_0} = P_0 \ln(x-x_0) \cancel{+c}$$

not needed, as it only gives a constant factor

for $x' \approx x_0$

$$w(x) \propto e^{-P_0 \ln(x-x_0)} = (x-x_0)^{-P_0}$$

using $\phi_1(x) \approx (x-x_0)^{\gamma_1}$:

$$\phi_2 \approx (x-x_0)^{\gamma_1} \int^x \frac{(x'-x_0)^{-P_0}}{(x'-x_0)^{2\gamma_1}} dx' = (x-x_0)^{\gamma_1} \int^x (x'-x_0)^{-P_0-2\gamma_1} dx'$$

but recall that $1 - P_0 = \gamma_1 + \gamma_2$, so $-P_0 - 2\gamma_1 = -1 - \underbrace{(\gamma_1 - \gamma_2)}_{N}$

so

$$\phi_2(x) \approx (x-x_0)^{\gamma_1} \int^x \frac{dx'}{(x'-x_0)^{N+1}} = \begin{cases} \dfrac{(x-x_0)^{\gamma_1 - N}}{-N} \propto (x-x_0)^{\gamma_2}, & N \neq 0 \\ (x-x_0)^{\gamma_1} \ln(x-x_0), & N = 0. \end{cases}$$

That is, for any γ_1 & γ_2

$$\phi_1(x) \approx (x-x_0)^{\gamma_1} \quad \text{for } x \text{ near } x_0$$

And, for any $\gamma_2 \neq \gamma_1$:

$$\phi_2(x) \approx (x-x_0)^{\gamma_2}$$

while for $\gamma_2 = \gamma_1$ — for x near x_0

$$\phi_2(x) \approx (x-x_0)^{\gamma_1} \ln(x-x_0)$$