Gradient

where $d\vec{r} = (dx, dy, dz)$

All independent and infinitesimal

$$\overrightarrow{\text{Grad}}\,\phi(\vec{r})\cdot d\vec{r} = \phi(\vec{r}+d\vec{r}) - \phi(\vec{r})$$

$$= \underbrace{\phi(x+dx,\ y+dy,\ z+dz)}_{\text{Taylor expand in last variable around } z} - \phi(x,y,z)$$

$$= \phi(x+dx, y+dy, z) + dz\frac{\partial\phi}{\partial z}(x+dx, y+dy, z) + \overbrace{\mathcal{O}(dz^2)}^{\text{very small}} - \phi(x,y,z)$$

expand both in second variable around y

$$= \phi(x+dx, y, z) + dy\frac{\partial\phi}{\partial y}(x+dx, y, z) + \mathcal{O}(dy^2) + dz\frac{\partial\phi}{\partial z}(x+dx, y, z) + \underbrace{\mathcal{O}(dy\,dz) + \mathcal{O}(dz^2)}_{} - \phi(x,y,z)$$

All very small.
Call them collectively $\mathcal{O}(dr^2)$

Now expand these three terms in the first variable around x.

$$= \cancel{\phi(x,y,z)} + dx\frac{\partial}{\partial x}\phi(x,y,z) + dy\frac{\partial}{\partial y}\phi(x,y,z) + dz\frac{\partial\phi}{\partial z}(x,y,z) + \mathcal{O}(dr^2) - \cancel{\phi(x,y,z)}$$

$$= d\vec{r}\cdot\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\phi(\vec{r}) = d\vec{r}\cdot\nabla\phi(\vec{r})$$

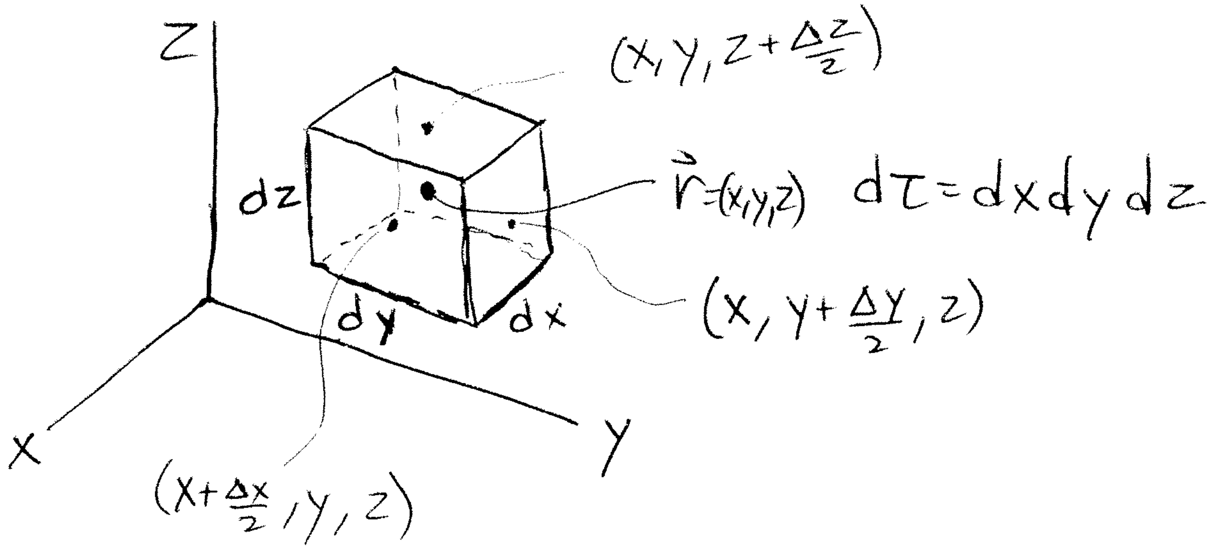
$$\therefore\ \overrightarrow{\text{Grad}}\,\phi(\vec{r})\cdot d\vec{r} = \nabla\phi(\vec{r})\cdot d\vec{r} \Longrightarrow \underline{\overrightarrow{\text{Grad}}\,\phi(\vec{r}) = \nabla\phi(\vec{r})}$$

because

Divergence

$$\mathrm{Div}\left[\vec{A}(\vec{r})\right] d\tau = \int_{S\text{ of }d\tau} \vec{A}\cdot d\vec{\sigma}$$

For simplicity, use an infinitesimal box



$$\int_{S\text{ of }d\tau} \vec{A}\cdot d\hat{\sigma} \approx \underbrace{\vec{A}\left(x,y,z+\tfrac{\Delta z}{2}\right)\cdot \underbrace{\left(dx\,dy\,\hat{z}\right)}_{d\vec{\sigma}}}_{\text{top}} + \underbrace{\vec{A}\left(x,y,z-\tfrac{\Delta z}{2}\right)\cdot\left(-dx\,dy\,\hat{z}\right)}_{\text{bottom}}$$

$$+ \underbrace{\vec{A}\left(x+\tfrac{\Delta x}{2},y,z\right)\cdot\left(dy\,dz\,\hat{x}\right)}_{\text{front face}} + \underbrace{\vec{A}\left(x-\tfrac{\Delta x}{2},y,z\right)\cdot\left(-dy\,dz\,\hat{x}\right)}_{\text{back face}}$$

$$+ \underbrace{\vec{A}\left(x,y+\tfrac{\Delta y}{2},z\right)\cdot\left(dx\,dz\,\hat{y}\right)}_{\text{right side}} + \underbrace{\vec{A}\left(x,y-\tfrac{\Delta y}{2},z\right)\cdot\left(-dx\,dz\,\hat{y}\right)}_{\text{left side}}$$

$$\mathrm{Div}\left[\hat{A}(\vec{r})\right] d\tau = \int_{S\,\mathrm{of}\,d\tau} \vec{A}\cdot d\vec{\sigma}$$

$$= dx\,dy \left[\underbrace{A_z\left(x,y,z+\tfrac{dz}{2}\right) - A_z\left(x,y,z-\tfrac{dz}{2}\right)}_{\frac{\partial A_z}{\partial z}(x,y,z)\,dz \;+\; \cancel{O(dz^2)}}\right]$$

$$+\, dy\,dz \left[\underbrace{A_x\left(x+\tfrac{dx}{2},y,z\right) - A_x\left(x-\tfrac{dx}{2},y,z\right)}_{\frac{\partial A_x}{\partial x}(x,y,z)\,dx}\right]$$

$$+\, dx\,dz \left[\underbrace{A_y\left(x,y+\tfrac{dy}{2},z\right) - A_y\left(x,y-\tfrac{dy}{2},z\right)}_{\frac{\partial A_y}{\partial y}(x,y,z)\,dy}\right]$$

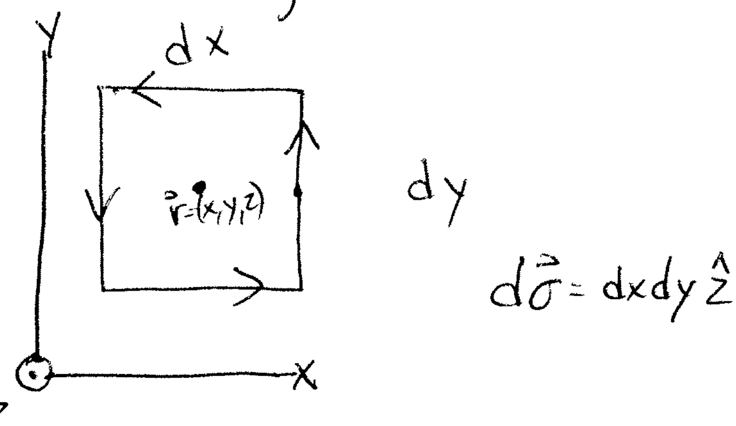
$$= \underbrace{dx\,dy\,dz}_{d\tau}\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\cdot \vec{A}(\vec{r}) = d\tau\; \nabla\cdot\vec{A}(\vec{r})$$

$$\therefore \quad \underline{\mathrm{Div}\left[\vec{A}(\vec{r})\right] = \nabla\cdot\vec{A}(\vec{r})}$$

Curl

$$\overrightarrow{\text{Curl}}\left[\vec{A}(\vec{r})\right]\cdot d\vec{\sigma} = \oint_{\partial \text{ of } d\vec{\sigma}} \vec{A}(\vec{r})\cdot d\vec{\lambda}$$

choose first a rectangular circuit



$$\oint_{\partial \text{ of } d\vec{\sigma}} \vec{A}(\vec{r})\cdot d\vec{\lambda} \approx \underbrace{\vec{A}\left(x+\frac{dx}{2},y,z\right)\cdot(dy\,\hat{y})}_{\text{right side}}$$

$$+ \underbrace{\vec{A}\left(x,y+\frac{dy}{2},z\right)\cdot(-dx\,\hat{x})}_{\text{top}} + \underbrace{\vec{A}\left(x-\frac{dx}{2},y,z\right)\cdot(-dy\,\hat{y})}_{\text{left}}$$

$$+ \underbrace{\vec{A}\left(x,y-\frac{dy}{2},z\right)\cdot(dx\,\hat{x})}_{\text{bottom}}$$

$$= dy\Big[\underbrace{A_y\left(x+\frac{dx}{2},y,z\right)-A_y\left(x-\frac{dx}{2},y,z\right)}_{\frac{\partial A_y}{\partial x}(\vec{r})\,dx\;\cancel{+\,\mathcal{O}(dx^2)}}\Big]$$

$$-dx\Big[\underbrace{A_x\left(x,y+\frac{dy}{2},z\right)-A_x\left(x,y-\frac{dy}{2},z\right)}_{\frac{\partial A_x}{\partial y}(\vec{r})\,dy}\Big]$$

$$\overrightarrow{\text{Curl}}\left[\vec{A}(\vec{r})\right]\cdot d\vec{\sigma} = \oint_{C} \vec{A}(\vec{r})\cdot d\vec{\lambda}$$

$$= dx\,dy\left[\frac{\partial A_y}{\partial x}(\vec{r}) - \frac{\partial A_x}{\partial y}(\vec{r})\right]$$

$$= \underbrace{dx\,dy\;\hat{z}}_{d\vec{\sigma}}\cdot\left[\nabla\times\vec{A}(\vec{r})\right]$$

Can do the same with circuits with normals in the $\hat{x}$ and $\hat{y}$ directions, to find

$$\underline{\overrightarrow{\text{Curl}}\left[\vec{A}(\vec{r})\right] = \nabla\times\vec{A}(\vec{r})}$$