PDE: $(\hat{\mathcal{L}}_r - \hat{\mathcal{L}}_t)\phi(\vec{r},t) = -\rho(\vec{r},t)$

ROI: $\vec{r}$ in V, $t > t_0$

IC: $\phi(\vec{r},t_0) = \phi_0(\vec{r})$, $\dot{\phi}(\vec{r},t_0) = \dot{\phi}_0(\vec{r})$

BC: $\phi\big|_{\text{surf. of } V}$ or $\hat{n}\cdot\nabla\phi\big|_{\text{surf. of } V}$ or Periodic

1

$\phi = \phi_h + \phi_p$

1.1

$(\hat{\mathcal{L}}_r - \hat{\mathcal{L}}_t)\phi_h = 0$

meets IC & BC

1.2

$(\hat{\mathcal{L}}_r - \hat{\mathcal{L}}_t)\phi_p = -\rho$

IC & BC = 0

$\phi_h = \phi_s(\vec{r}) + \phi_T(\vec{r},t)$

1.1.1

$\hat{\mathcal{L}}_r\phi_s = 0$

meets constant BC

(not used for periodic BC)

1.1.2

$(\hat{\mathcal{L}}_r - \hat{\mathcal{L}}_t)\phi_T = 0$

BC = 0

IC: $\phi(\vec{r},t_0) = \phi_0 - \phi_s$, $\dot{\phi}(\vec{r},t_0) = \dot{\phi}_0$

$\phi_T = \sum_n U_n(\vec{r})T_n(t)$

1.1.2.1

$(\hat{\mathcal{L}}_r - \lambda_n)U_n(\vec{r}) = 0$

(Stürm-Liouville)

B.C. = 0

together meet IC. of ϕ_T

1.1.2.2

$(\hat{\mathcal{L}}_t - \lambda_n)T_n(t) = 0$

Spatial Sep. of Variables

$U_{lmn}(\vec{r}) = \sum_{lmn} Q_{1l}(q_1)Q_2(q_2)Q_3(q_3)$

Solving 1.1.2.2 in special cases

$$\left(\hat{\mathcal{L}}_t - \lambda_n\right) T_n(t) = 0, \qquad \text{D.E.}$$

where λ_n is dictated by 1.1.2.1

R.O.I.

$t > t_0$

$T_n(t_0)$ such that ϕ_T meets initial conditions.

Note: This is not a Stürm-Liouville problem, because it has I.C.

- Typical case 1: (Diffusion)

$$\hat{\mathcal{L}}_t = D \frac{d}{dt}$$

$$D \frac{d T_n}{dt} - \lambda_n T_n = 0$$

$$T_n(t) = \underline{A_n e^{\frac{\lambda_n t}{D}}}$$

T_n with <u>largest λ_n will dominate.</u>

- Typical case 2: (damped oscillation)

$$\hat{\mathcal{L}}_t = \frac{1}{V^2} \frac{d^2}{dt^2} + \frac{2\gamma}{V^2} \frac{d}{dt}$$

$$\frac{1}{V^2} \ddot{T}_n + \frac{2\gamma}{V^2} \dot{T}_n - \lambda_n T_n = 0$$

propose $T_n(t) = A_n e^{\alpha t}$

$$\frac{\alpha^2}{V^2} + 2\frac{\gamma}{V^2}\alpha - \lambda_n = 0$$

$$\alpha_\pm = -\gamma \pm \sqrt{\gamma^2 + \lambda_n V^2}$$

Three cases:

$$T_n(t) = \begin{cases} A_n e^{-\left(\gamma - \sqrt{\gamma^2+\lambda_n V^2}\right)t} + B_n e^{-\left(\gamma + \sqrt{\gamma^2+\lambda_n V^2}\right)t}, & \gamma^2 + \lambda_n V^2 > 0 \text{ (overdamped)} \\ A_n e^{-\gamma t} + \underline{B_n t e^{-\gamma t}}, & \gamma^2 + \lambda_n V^2 = 0 \text{ (critically damped)} \\ A_n e^{-\gamma t} \cos\left(\sqrt{|\gamma^2+\lambda_n V^2|}\, t\right) + B_n e^{-\gamma t} \sin\left(\sqrt{|\gamma^2+\lambda_n V^2|}\, t\right), & \gamma^2 + \lambda_n V^2 < 0 \text{ (underdamped)} \end{cases}$$

(found using $\phi_2(t) = \phi_1 \int^t \frac{W(t')}{\phi_1^2(t')} dt'$)

For the overdamped case, the first part of the solution with the largest λ_n will dominate for large t.

This component of the solution is called the ground state.

Solving (1.1.2.1) (and (1.1.1)) in special cases

Consider $\hat{\mathcal{L}}_r = \nabla^2 \implies \lambda = -k^2 < 0$

(1.1.2.1) becomes Helmholtz equation

D.E. $(\nabla^2 + k^2)\, U(\vec{r}) = 0$

R.O.I $\vec{r}$ in V

B.C. $U\Big|_{\text{surface of } V} = 0$ (Dirichlet)

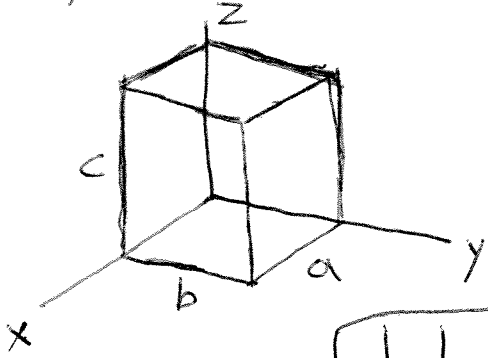
Use separation of variables. This works in 11 types of coordinate systems:

Cartesian	Confocal ellipsoidal	Elliptic cylindrical
Cylindrical	Confocal paraboloidal	Parabolic cylindrical
Spherical	Conical	Oblate Spheroidal
	Paraboloidal	Prolate spheroidal

We will only discuss the typical 3:

Cartesian, Cylindrical, spherical.

1) Cartesian



$$\boxed{U(\vec{r}) = X(x)\, Y(y)\, Z(z)}$$

so $\quad X(0) = X(a) = 0$
$\quad\quad Y(0) = Y(b) = 0$
$\quad\quad Z(0) = Z(c) = 0$

1.1.2.1

$$0 = \nabla^2 U + k^2 U = X'' Y Z + X Y'' Z + X Y Z'' + k^2 X Y Z$$

divide by XYZ:

$$\underbrace{\frac{X''}{X}}_{-k_x^2} + \underbrace{\frac{Y''}{Y}}_{-k_y^2} + \underbrace{\frac{Z''}{Z}}_{-k_z^2} + k^2 = 0$$

for X:

$$X'' + k_x^2 X = 0, \quad X(0) = X(a) = 0 \Rightarrow X_l(x) = \sin\left(\frac{\pi l x}{a}\right)$$

with $k_x = \frac{\pi l}{a}$, $l = 1, 2, \ldots$

Same for Y, Z, so

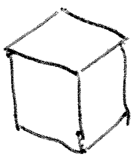
$$\boxed{U_{lmn}(\vec{r}) = \sin\left(\frac{\pi l x}{a}\right) \sin\left(\frac{\pi m y}{b}\right) \sin\left(\frac{\pi n z}{c}\right)}$$

$$\lambda_{lmn} = -k_{lmn}^2 = -(k_x^2 + k_y^2 + k_z^2) = -\pi^2\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)$$

for $l, m, n = 1, 2, \ldots$

Notice: degeneracy if $\frac{a}{b}$ or $\frac{b}{c}$ or $\frac{a}{c}$ are rationals.

Ground state: $\qquad \underline{\lambda_{111} = -\pi^2\left(a^{-2}+b^{-2}+c^{-2}\right)}$



Can solve (1.1.1) using a similar technique.

$$\mathcal{L}_r\phi_s = \nabla^2\phi_s(\vec{r}) = 0, \quad \phi_s\Big|_{\text{surf. of } V} = \text{specified}$$

Again, try $\phi_s = X\,Y\,Z$. This leads to

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

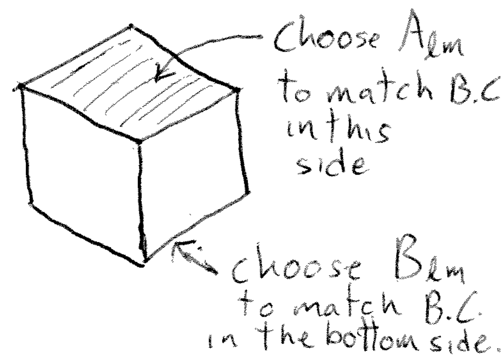
Choose first $\quad X = X_l = \sin\left(\frac{\pi l x}{a}\right), \; Y = Y_m = \sin\left(\frac{\pi m y}{b}\right)$

$$-\left(\frac{\pi l}{a}\right)^2 - \left(\frac{\pi m}{b}\right)^2 + \frac{Z''}{Z} = 0 \Rightarrow Z'' = \pi^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}\right) Z$$

$$Z = Z_{lm} \quad \text{where}$$

$$Z_{lm}(z) = \underset{\text{Vanishes at } z=0}{A_{lm}\sinh\left[\pi\sqrt{\frac{l^2}{a^2}+\frac{m^2}{b^2}}\,z\right]} + \underset{\text{Vanishes at } z=c}{B_{lm}\sinh\left[\pi\sqrt{\frac{l^2}{a^2}+\frac{m^2}{b^2}}\,(c-z)\right]}$$

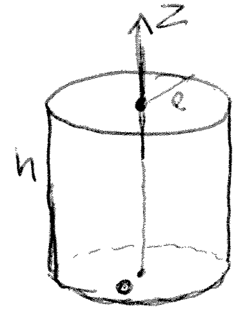
$$\phi_s^{(z)}(\vec{r}) = \sum_{l,m=1}^{\infty} X_l\, Y_m\, Z_{lm}$$



Same for x and y, so

$$\phi_s = \phi_s^{(x)} + \phi_s^{(y)} + \phi_s^{(z)}$$

Case 2) Cylindrical: $\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$



First solve (1.1.2.1):

$$\nabla^2 U + k^2 U = 0, \quad U\big|_{\text{boundary}} = 0$$

Propose $U(\vec{r}) = R(r)\,\Theta(\theta)\,Z(z)$

$$\frac{\nabla^2 U}{U} + k^2 = \frac{\frac{1}{r}(rR')'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{Z''}{Z} + k^2 = 0$$

these two must be constants.

because $Z(0) = Z(h) = 0$,
it clearly is given by $\underline{Z_\ell(z) = \sin\left(\frac{\pi \ell z}{h}\right)}$

so $\frac{Z''}{Z} = -\left(\frac{\pi \ell}{h}\right)^2$

Also, because Θ must be periodic, i.e. $\Theta(\theta + 2\pi) \overset{\theta}{\equiv} \Theta(\theta)$
it is given by

$$\underline{\Theta_m(\theta) = A_m \cos(m\theta) + B_m \sin(m\theta)}$$

so $\frac{\Theta''}{\Theta} = -m^2$

The equation for R is then

$$\frac{\frac{1}{r}(rR')'}{R} - \frac{m^2}{r^2} + k^2 - \left(\frac{\pi \ell}{h}\right)^2 = 0$$

Can be rewritten as:

$$\boxed{R'' + \frac{R'}{r} + \left(\alpha^2 - \frac{m^2}{r^2}\right) R = 0}, \text{ where } \alpha^2 = k^2 - \left(\frac{\pi l}{h}\right)^2$$

Bessel's equation.

Solutions: if $\alpha = 0$:

$$R'' + \frac{R'}{r} - m^2 \frac{R}{r^2} = 0 \quad , \text{ try } R = r^\gamma$$

$$\gamma(\gamma-1)\cancel{r^{\gamma-2}} + \gamma \cancel{r^{\gamma-2}} - m^2 \cancel{r^{\gamma-2}} = 0$$

$$\gamma^2 - m^2 = 0 \quad \therefore \quad \gamma_{\frac{1}{2}} = \pm m \quad , \quad m \neq 0$$

$$R = A r^m + B r^{-m}$$

If $m=0$ one solution is $R = A$.
can find second solution:

$$p(r) = \frac{1}{r} \quad \therefore \quad P(r) = \int^r p(r')dr' = \ln r \;,\; w = e^{-P} = \frac{1}{r}$$

$$R_2 = R_1(r) \int^r \frac{w(r')}{R_1^2(r')} dr' = \frac{1}{A} \int^r \frac{dr'}{r'} = \frac{1}{A} \ln r$$

so $R = A + B \ln r$

$\alpha \neq 0$: $\qquad R'' + \frac{R'}{r} + \left(\alpha^2 - \frac{m^2}{r^2}\right) R = 0$

$p(r) = \frac{1}{r}$, $q(r) = \frac{\alpha^2 r^2 - m^2}{r^2}$, so $r=0$ is a regular singular point

Can use Frobenius:

$\cdot\ R(r) = A\, r^{\gamma} \sum_{n=0}^{\infty} a_n r^n$, $a_0 = 1$

$r\,p(r) = \sum_{n=0}^{\infty} P_n r^n = 1$, so $P_0 = 1$, $P_{n>0} = 0$.

$r^2 q(r) = \sum_{n=0}^{\infty} Q_n r^n = -m^2 + \alpha^2 r^2$, so $Q_0 = -m^2$, $Q_1 = 0$
$Q_2 = \alpha^2$, $Q_{n>2} = 0$.

Indicial equation: $\gamma(\gamma-1) + \gamma P_0 + Q_0 = 0$

$\gamma_{1,2} = \pm m$ differ by an integer.
to find second solution, need to use a series with a log.

Solutions are Bessel functions:

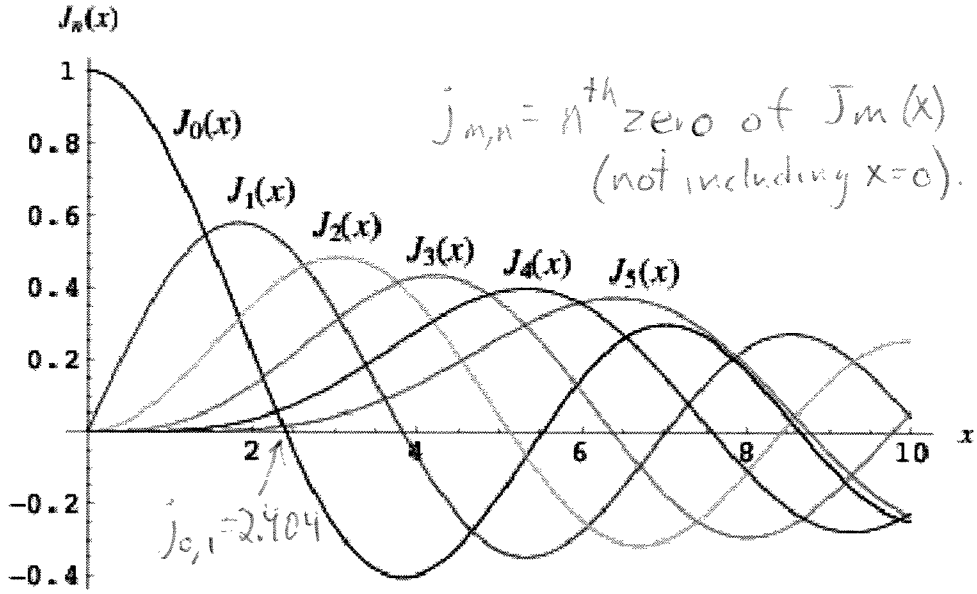
$$R(r) = A\, J_m(\alpha r) + B\, Y_m(\alpha r)$$

Bessel function of the 1st type. (1st solution) According to indicial equation, behaves as r^m for small r. As shown in homework, oscillates and decays as $\frac{1}{\sqrt{r}}$ for large r.	Bessel function of the 2nd type (2nd solution) for $m=0$, blows up like $\ln r$, for $m>0$, blows up like r^{-m} for small r. Also decays and oscillates.

Bessel Functions

1st Kind

$j_{m,n}$ = n^{th} zero of $J_m(x)$ (not including $x=0$).

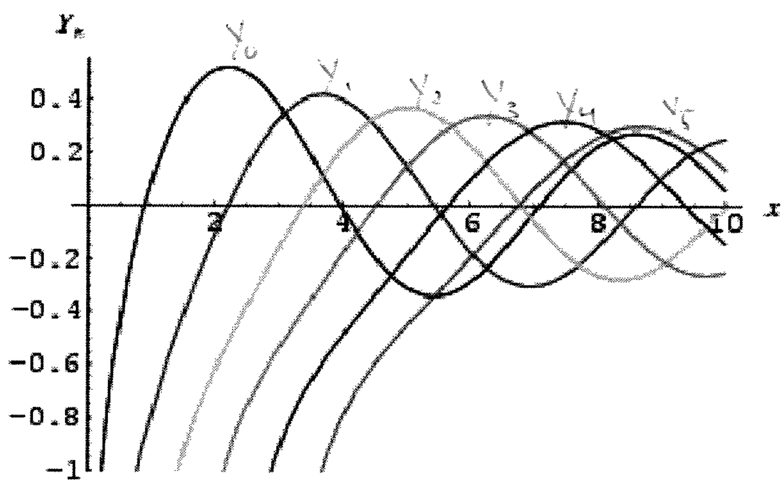


$$J_m = x^m \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+m}\, n!\,(m+n)!}\, x^{2n}$$

$$J_m = \frac{1}{2\pi i^m} \int_0^{2\pi} e^{ix\cos\phi + im\phi}\, d\phi$$

$$2m J_m = x\left(J_{m-1} + J_{m+1}\right)$$

2nd Kind



so
$$R(r)=\begin{cases} A + B\ln r, & \alpha=0,\ m=0 \\ Ar^{m} + Br^{-m}, & \alpha=0,\ m\neq 0 \\ A J_m(\alpha r) + B\, Y_m(\alpha r), & \alpha\neq 0 \end{cases}$$

Because our R.O.I. includes $r=0$, we set $B=0$.

Also, we need $R(\rho)=0$, so α must be $\neq 0$.

$$R_m(r) = J_m(\alpha r)$$

$$R_m(\rho) = J_m(\alpha\rho) = 0 \Rightarrow \alpha = \frac{j_{m,n}}{\rho}$$

$$\underline{R_{m,n}(r) = J_m\left(\frac{j_{m,n}}{\rho} r\right)}$$

so
$$\boxed{U_{lmn} = J_m\left(\frac{j_{m,n}}{\rho} r\right)\left[A_{lmn}\cos(m\theta) + B_{lmn}\sin(m\theta)\right]\sin\left(\frac{\pi l z}{h}\right)}$$

$$\lambda_{lmn} = -k^2 = -\left[\left(\frac{\pi l}{h}\right)^2 + \left(\frac{j_{m,n}}{\rho}\right)^2\right]$$

$$m = 0,1,2,\ldots \quad , \quad l,n = 1,2,3,\ldots$$

Ground state: $\lambda_{101} = -\left[\left(\frac{\pi}{h}\right)^2 + \left(\frac{j_{0,1}}{\rho}\right)^2\right]$

$j_{0,1} = 2.404\ldots$

Now, let's solve for (1.1.1) for : $\nabla^2 \phi_s(\vec{r}) = 0$

Let $\phi_s = \phi_s^{(z)} + \phi_s^{(r)}$, where:

$\phi_s^{(z)}$ meets B.C. at the caps, and vanishes at the sides 

$\phi_s^{(r)}$ meets B.C at the sides, and vanishes at the caps 

First, for $\phi_s^{(z)}$: use $\phi_s^{(z)} = R_{m,n} \Theta_m Z$

with:

$$R_{m,n} = J_m\left(\frac{j_{m,n}}{\rho} r\right), \quad \Theta_m = A\cos m\theta + B\sin m\theta$$

then:

$$\frac{\nabla^2 \phi_s^{(z)}}{\phi_s^{(z)}} = \frac{R''_{m,n} + r R'_{m,n}}{R_{m,n}} + \frac{1}{r^2}\frac{\Theta''_m}{\Theta_m} + \frac{Z''}{Z} = \frac{Z''}{Z} - \left[\left(\frac{j_{m,n}}{\rho}\right)^2 - \cancel{\frac{m^2}{r^2}}\right] - \cancel{\frac{m^2}{r^2}} = 0$$

so $Z'' = \left(\frac{j_{m,n}}{\rho}\right)^2 Z$

$$Z_{m,n} = a\sinh\left(\frac{j_{m,n}}{\rho} z\right) + b\sinh\left(\frac{j_{m,n}}{\rho}(h-z)\right)$$

putting it all together:

$$\phi_s^{(z)} = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[J_m\left(\frac{j_{m,n}}{\rho} r\right)\sinh\left(\frac{j_{m,n}}{\rho} z\right)\left(A^{(1)}_{m,n}\cos m\theta + B^{(1)}_{m,n}\sin m\theta\right) + J_m\left(\frac{j_{m,n}}{\rho} r\right)\sinh\left(\frac{j_{m,n}}{\rho}(c-z)\right)\left(A^{(2)}_{m,n}\cos m\theta + B^{(2)}_{m,n}\sin m\theta\right)\right]$$

Now, for $\phi_s^{(r)}$, use $\phi_s^{(r)} = R\,\Theta_m Z_\ell$

with: $\Theta_m = A\cos m\theta + B\sin m\theta$, $Z_\ell = \sin\left(\frac{\pi \ell z}{h}\right)$
so we get

$$R'' + \frac{R'}{r} + \left(-\left(\frac{\pi \ell}{h}\right)^2 - \frac{m^2}{r^2}\right)R = 0$$

notice sign.

Solutions: modified Bessel functions

$$R = A\,I_m\left(\frac{\pi \ell}{h} r\right) + B\,K_m\left(\frac{\pi \ell}{h} r\right)$$

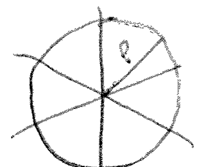
I_m is the "analytic continuation" into the imaginary axis of J_m: $I_m(x) = i^{-m} J_m(ix)$. It also proportiona[l] to r^m for small r (because it satisfies the same indicial equation). It does <u>not</u> oscillate, and grows like an exponential.

K_m diverges at zero like $\ln r$ for $m=0$ and like r^{-m} for $m \neq 0$. It decays like an exponential. We discard it if the problem includes $r=0$.

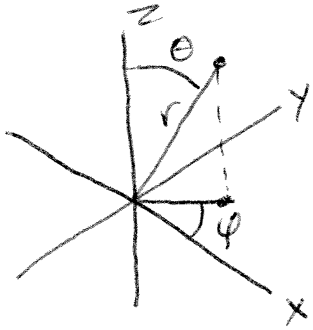
$$R_{\ell m} = I_m\left(\frac{\pi \ell}{h} r\right)$$

$$\phi_s^{(r)} = \sum_{m=0}^{\infty}\sum_{\ell=1}^{\infty} I_m\left(\frac{\pi \ell}{h} r\right)\left[A_{\ell m}\cos m\theta + B_{\ell m}\sin m\theta\right]\sin\left(\frac{\pi \ell z}{h}\right).$$

3) Spherical



$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$$



Try $U(\vec{r}) = R(r)\,\Theta(\theta)\,\Phi(\varphi)$

where

$R(0)$ = finite, $R(\rho) = 0$ (analytic)

$\Theta(\theta)$ = finite

$\Phi(\varphi + 2\pi) \overset{\varphi}{\equiv} \Phi(\varphi)$.

1.1.2.1 $$\frac{\nabla^2 U}{U} + k^2 = \underbrace{\frac{1}{r^2}\left(r^2 R'\right)'}_{R} + \frac{1}{r^2}\underbrace{\frac{1}{\sin\theta}\left(\sin\theta\,\Theta'\right)'}_{\Theta} + \frac{1}{r^2\sin^2\theta}\frac{\Phi''}{\Phi} + k^2 = 0$$

clearly $\Phi = \Phi_m(\varphi) = A\cos m\varphi + B\sin m\varphi$

$$\frac{R'' + 2R'/r}{R} + \frac{1}{r^2}\left(\frac{\Theta'' + \frac{\cos\theta}{\sin\theta}\Theta'}{\Theta}\right) + k^2 - \frac{m^2}{r^2\sin^2\theta} = 0$$

or:

$$\frac{R'' + 2R'/r}{R} + \frac{1}{r^2}\underbrace{\left(\frac{\Theta'' + \frac{\cos\theta}{\sin\theta}\Theta'}{\Theta} - \frac{m^2}{\sin^2\theta}\right)}_{-\beta} + k^2 = 0$$

so:

$$R'' + \frac{2R'}{r} + \left(k^2 - \frac{\beta}{r^2}\right)R = 0, \quad \Theta'' + \frac{\cos\theta}{\sin\theta}\Theta' + \left(\beta - \frac{m^2}{\sin^2\theta}\right)\Theta = 0$$

Do radial first: $R'' + \frac{2R'}{r} + \left(k^2 - \frac{\beta}{r^2}\right) R = 0$

Looks like Bessel's eq. except for this two

can remove it using $R = \frac{u}{\sqrt{r}}$,

$$R' = \frac{u'}{\sqrt{r}} - \frac{1}{2}\frac{u}{r^{3/2}}, \quad R'' = \frac{u''}{r^{1/2}} - \frac{u'}{r^{3/2}} + \frac{3}{4}\frac{u}{r^{5/2}}$$

$$\frac{1}{\sqrt{r}}\left[u'' - \frac{u'}{r} + \frac{3}{4}\frac{u}{r^2} + \frac{2}{r}\left(u' - \frac{u}{2r}\right) + \left(k^2 - \frac{\beta}{r^2}\right)u\right] = 0$$

$$u'' + \frac{u'}{r} + \left(k^2 - \frac{\beta + \frac{1}{4}}{r^2}\right)u = 0$$

Again, $r=0$ is a regular singular point.

$$p(r) = \frac{1}{r} \Rightarrow P_0 = 1, \quad q(r) = k^2 - \frac{\beta + \frac{1}{4}}{r^2} \Rightarrow Q_0 = -\left(\beta + \tfrac{1}{4}\right)$$

indicial Eq. $\gamma(\gamma - 1) + P_0\gamma + Q_0 = \gamma^2 - \left(\beta + \frac{1}{4}\right) = 0$

$\gamma_{\frac{1}{2}} = \pm\sqrt{\beta + \frac{1}{4}}$, so, for small r, $u_1 \propto r^{\sqrt{\beta + \frac{1}{4}}}$,

$R_1 = \frac{u_1}{\sqrt{r}} \propto r^{\sqrt{\beta + \frac{1}{4}} - \frac{1}{2}}$, but we want $R_1 \propto r^{\ell}$ ← non-negative integer

so $\ell = \sqrt{\beta + \frac{1}{4}} - \frac{1}{2} \Rightarrow \beta = \left(\ell + \frac{1}{2}\right)^2 - \frac{1}{4} = \underline{\ell(\ell+1)}$

and

$$u'' + \frac{u'}{r} + \left(k^2 - \frac{\left(\ell + \frac{1}{2}\right)^2}{r^2}\right)u = 0$$

$$u_\ell = A\, J_{\ell + \frac{1}{2}}(kr) + \cancel{B\, Y_{\ell + \frac{1}{2}}(kr)}$$

because it diverges as $r^{-\ell-1}$

$\ell = 0, 1, \ldots$

Notice, for $l=0$ $\qquad u'' + \frac{u'}{r} + \left(k^2 + \frac{1}{4r^2}\right)u = 0$

try $u = \frac{v}{\sqrt{r}} \Rightarrow \frac{1}{\cancel{\sqrt{r}}}\left(v'' + k^2 v\right) = 0$

$v = a\sin kr + b\cos kr$

$$R = \frac{u}{\sqrt{r}} = \frac{v}{r} = a\frac{\sin kr}{r} + \cancel{b\frac{\cos kr}{r}}$$

in fact $J_{1/2}(kr) = \sqrt{\frac{2}{\pi}}\,\frac{\sin kr}{\sqrt{kr}}$.

Boundary Conditions: $R(\rho) = \frac{J_{l+\frac{1}{2}}(k\rho)}{\sqrt{\rho}} = 0$

$k_{l,n} = \frac{j_{l,n}}{\rho}$, $\quad j_{l,n} = n^{th}$ zero of $J_{l+\frac{1}{2}}$.

f. ex. $j_{0,1} = \pi$

$$\underline{R_{l,n} = \frac{J_{l+\frac{1}{2}}\left(\frac{j_{l,n}}{\rho} r\right)}{\sqrt{r}}}$$

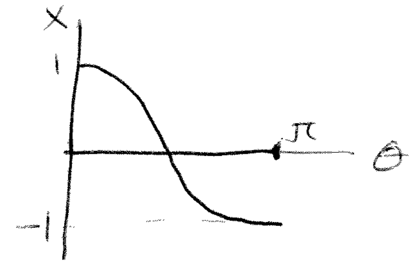
Notice, we haven't done (H) yet, but we already know

$$\lambda_{l,n} = -k_{l,n}^2 = -\left(\frac{j_{l,n}}{\rho}\right)^2$$

Ground state: $\underline{\lambda_{0,1} = -\frac{\pi^2}{\rho^2}}$

Finally, $\Theta'' + \frac{\cos\theta}{\sin\theta}\Theta' + \left(\overset{\ell(\ell+1)}{\beta} - \frac{m^2}{\sin^2\theta}\right)\Theta = 0$, $0 \le \theta \le \pi$

change variables to $x = \cos\theta$



$\Theta(\theta) = P(\cos\theta)$
$\Theta'(\theta) = -\sin\theta\, P'$, so $\frac{\Theta'}{\sin\theta} = -P'$

$\Theta''(\theta) = -\cos\theta\, P' + \sin^2\theta\, P'' = (1-x^2)\, P'' - x P'$

$$(1-x^2)P'' - xP' - xP' + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right] P = 0$$

$$(1-x^2)\, P'' - 2x P' + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right] P = 0$$ Legendre's Eq.

can write as $P'' + p(x)\, P' + q(x)\, P = 0$
with

$$p(x) = -\frac{2x}{1-x^2} , \quad q(x) = \frac{\ell(\ell+1)}{1-x^2} - \frac{m^2}{(1-x^2)^2}$$

regular singular points at $x = \pm 1$
Take, for example, $x = -1$.

$$(x+1)p(x) = \frac{-2x}{1-x} = \sum_{n=0}^{\infty} P_n (x+1)^n \Rightarrow P_0 = \left.\frac{-2x}{1-x}\right|_{-1} = 1$$

$$(x+1)^2 q(x) = \frac{\ell(\ell+1)(x+1)}{1-x} - \frac{m^2}{(1-x)^2} = \sum_{n=0}^{\infty} Q_n (x+1)^n \Rightarrow Q_0 = -\frac{m^2}{4}$$

indicial: $\gamma(\gamma-1) + P_0\gamma + Q_0 = \gamma^2 - \frac{m^2}{4} \Rightarrow \gamma_{1,2} = \pm\frac{m}{2}$

$\gamma_1 - \gamma_2$ is integer, so need prescription for 2^{nd} solution.
It diverges, however, at $x = \pm 1$, so we ignore it.

By doing Frobenius, can find solutions

$$P_\ell^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x) \quad \text{Associated Legendre Polynomials}$$

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2-1)^\ell \quad \text{Legendre Polynomials}$$

So, putting everything together:

$$U_{\ell mn}(\vec{r}) = \frac{J_{\ell+\frac{1}{2}}\left(\frac{j_{\ell,n}}{\ell} r\right)}{\sqrt{r}} \underbrace{\left[A_{\ell mn} \cos m\varphi + B_{\ell mn} \sin m\varphi\right] P_\ell^m(\cos\theta)}$$

can combine into spherical harmonics:

$$Y_\ell^m(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos\theta) e^{im\varphi}$$

Now, do (1.1.1): $\nabla^2 \phi_s = 0$. $\quad \phi_s \big|_{\text{surface of sphere}} = \text{prescribed}$.

choose $\phi_s = R\, \Theta_{\ell m} \Phi_m$, where $\Theta_{\ell m} = P_\ell^m(\cos\theta)$,

$\Phi_m = A \cos m\varphi + B \sin m\varphi$.

because $k=0$:

$$R'' + \frac{2R'}{r} - \frac{\beta}{r^2} R = 0.$$

Try $R = r^\gamma$: $\cancel{r^{\gamma-2}}\left[\gamma(\gamma-1) + 2\gamma - \beta\right] = 0$, $\gamma(\gamma+1) = \beta = \ell(\ell+1)$

$$\gamma = \begin{cases} \ell \\ -\ell-1 \end{cases} \qquad R(r) = A r^\ell + \cancel{B r^{-\ell-1}}$$

$$\phi_s(\vec{r}) = \sum_{\ell,m=0}^{\infty} r^\ell \left[A_{\ell m} \cos m\varphi + B_{\ell m} \sin m\varphi\right] P_\ell^m(\cos\theta)$$

Finally, consider solving (1.2):

$$\left(\hat{\mathcal{L}}_r - \hat{\mathcal{L}}_t\right)\phi_p = -\rho(\vec{r},t) \qquad \text{D.E.}$$

$$\vec{r} \text{ in } V, \quad t > t_0 \qquad \text{R.O.I.}$$

$$\phi_p(\vec{r},t_0) = 0, \quad \dot{\phi}_p(\vec{r},t_0) = 0 \qquad \text{I.C.}$$

for 2^{nd} order $\hat{\mathcal{L}}_t$

$$\phi_p(\vec{r},t)\Big|_{surf} = 0 \quad \text{or} \quad \hat{n}\cdot\nabla\phi_p(\vec{r},t)\Big|_{surf} = 0 \quad \text{or periodic B.C.}$$

Let us consider the case

$$\hat{\mathcal{L}}_r = \nabla^2, \quad \hat{\mathcal{L}}_t = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}$$

Fourier transform in time:

$$\hat{\mathcal{F}}_{t\to\omega}\left[\left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\phi_p(\vec{r},t)\right] = \hat{\mathcal{F}}_{t\to\omega}\left[-\rho(\vec{r},t)\right]$$

or

$$\left(\nabla^2 + \frac{\omega^2}{v^2}\right)\tilde{\phi}_p(\vec{r},\omega) = -\tilde{\rho}(\vec{r},\omega)$$

Inhomogeneous Helmholtz Equation.

Use Green's functions:

If $\tilde{\phi}_p$ is the response to $\tilde{\rho}$, i.e.

$$\tilde{\rho} \longrightarrow \tilde{\phi}_p\,,$$

then we can write

$$\tilde{\rho}(\vec{r},\omega) = \int_V \tilde{\rho}(\vec{r}\,';\omega)\, \delta(\vec{r}-\vec{r}\,')\, d^3r'\,,$$

so that

$$\tilde{\phi}_p(\vec{r},\omega) = \int_V \tilde{\rho}(\vec{r}\,';\omega)\, \tilde{G}(\vec{r},\vec{r}\,';\omega)\, d^3r'\,,$$

where $\tilde{G}(\vec{r},\vec{r}\,';\omega)$ is the response to $\delta(\vec{r}-\vec{r}\,')$

This means that

$$\boxed{\begin{array}{ll} \left(\nabla^2 + \frac{\omega^2}{V^2}\right) \tilde{G}(\vec{r},\vec{r}\,';\omega) = -\,\delta(\vec{r}-\vec{r}\,') & \text{D.E.} \\ \vec{r} \text{ in } V & \text{R.O.I} \\ \tilde{G}(\vec{r},\vec{r}\,';\omega) \text{ vanishes at prescribed B.C.} & \end{array}}$$

- Let us consider the case when
  $V =$ <u>all space</u>.

Due to shift invariance

$$\tilde{G}(\vec{r},\vec{r}\,';\omega) = \mathcal{G}(\vec{r}-\vec{r}\,';\omega)$$

$$\Big(\nabla^2 + \underbrace{\frac{\omega^2}{V^2}}_{k^2}\Big)\mathcal{G}(\vec{r};\omega) = -\,\delta(\vec{r})$$

Due to rotation invariance

$$\mathcal{G}(\vec{r},\omega) = g(|\vec{r}|,\omega) = g(r,\omega)$$

$$\nabla^2 g(|\vec{r}|,\omega) = \frac{\partial^2}{\partial r^2} g(r,\omega) + \frac{2}{r}\frac{\partial g}{\partial r}(r,\omega), \text{ so}$$

$$g'' + \frac{2}{r}g' + k^2 g = 0 \quad \text{for } r>0.$$

As we saw earlier, this can be written as $(rg)'' + k^2 rg = 0$.

so $$g(r,\omega) = A\,\underbrace{\frac{\cos kr}{r}}_{\text{singular}} + B\,\underbrace{\frac{\sin kr}{r}}_{\text{well-behaved}}$$

both $\to 0$ as $r \to \infty$.

We now find A & B from matching conditions at $r=0$.

Integrating over a sphere of radius ε:

$$\int_{S_\varepsilon} \left(\nabla^2 + \frac{\omega^2}{v^2}\right) g(|\vec{r}|,\omega)\, d^3r = \int_{S_\varepsilon} \left(-\delta(\vec{r})\right) d^3r = -1$$

consider separately both terms

$$\int_{S_\varepsilon} \nabla^2 g(|\vec{r}|,\omega)\, d^3r \underset{\text{Gauss theorem}}{=} \int_{S \text{ of } S_\varepsilon} \nabla g(|\vec{r}|,\omega)\cdot d\vec{\sigma}$$

$$= \int_{S \text{ of } S_\varepsilon} \frac{\partial}{\partial r} g(r,\omega)\, d\sigma = 4\pi\varepsilon^2\, g'(\varepsilon,\omega)$$

$$= 4\pi\varepsilon^2 \left[-Ak\frac{\sin kr}{r} - A\frac{\cos kr}{r^2} + Bk\frac{\cos kr}{r} - B\frac{\sin kr}{r^2} \right]\Bigg|_\varepsilon$$

$$\longrightarrow \underline{-4\pi A} \quad \text{as } \varepsilon \to 0.$$

$$\int_{S_\varepsilon} \frac{\omega^2}{V^2} g(|\vec{r}|,\omega)\, d^3r = \frac{\omega^2}{V^2} 4\pi \int_0^\varepsilon r^{\cancel{2}} \left[A\frac{\cos kr}{\cancel{r}} + B\frac{\sin kr}{\cancel{r}} \right] dr$$

$$\longrightarrow 0 \quad \text{as } \varepsilon \to 0.$$

Therefore

$$-4\pi A = -1, \quad A = \frac{1}{4\pi}$$

No conditions on B, because

$$B\frac{\sin(k|\vec{r}-\vec{r}'|)}{|\vec{r}-\vec{r}'|} \text{ is a mode!}$$

In summary

$$g(r,\omega) = \frac{\cos(kr)}{4\pi r} + B\,\frac{\sin(kr)}{r},$$

$$\tilde{G}(\vec{r},\vec{r}\,';\omega) = \frac{\cos\left(\frac{\omega}{v}|\vec{r}-\vec{r}\,'|\right)}{4\pi|\vec{r}-\vec{r}\,'|} + B\,\frac{\sin\left(\frac{\omega}{v}|\vec{r}-\vec{r}\,'|\right)}{|\vec{r}-\vec{r}\,'|}$$

<u>Note</u>: In diffraction theory one chooses

$$B = \frac{i}{4\pi}$$

so $$\tilde{G}(\vec{r},\vec{r}\,';\omega) = \frac{e^{i\frac{\omega}{v}|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|}$$

This is so $\tilde{G}$ satisfies the so-called "Sommerfeld Radiation Condition". The reason for this will become apparent later.

Note 2: Notice that the above Green's functions correspond to Helmoltz's equation. If instead we are dealing with Laplace's equation:

$$\nabla^2\phi = 0,$$

the Green's function results from letting $\omega \to 0$:

$$\hat{G}(\vec{r},\vec{r}\,') = \frac{1}{4\pi|\vec{r}-\vec{r}\,'|}.$$

The mode disappears.

Finally, we must inverse-Fourier transform to find

$G(\vec{r},\vec{r}',t)$, such that

$$\left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right) G(\vec{r},\vec{r}',t) = -\delta(\vec{r}-\vec{r}')\,\delta(t).$$

$$G(\vec{r},\vec{r}',t) = \hat{\mathcal{F}}^{-1}_{\omega\to t}\,\tilde{G}(\vec{r},\vec{r}';\omega)$$

$$= \frac{1}{2|\vec{r}-\vec{r}'|}\left\{\left(\frac{1}{4\pi}+\frac{B}{i}\right)\delta\left(t-\frac{|\vec{r}-\vec{r}'|}{v}\right)\right. \quad \text{retarded}$$

$$\left. + \left(\frac{1}{4\pi}-\frac{B}{i}\right)\delta\left(t+\frac{|\vec{r}-\vec{r}'|}{v}\right)\right\} \quad \text{advanced.}$$

Due to causality, we must cancel the "advanced" part, so we choose $B = \frac{i}{4\pi}$.

$$\underline{G(\vec{r},\vec{r}',t) = \frac{1}{4\pi|\vec{r}-\vec{r}'|}\,\delta\left(t-\frac{|\vec{r}-\vec{r}'|}{v}\right)}$$

Therefore

$$\phi_\rho(\vec{r},t) = \int_{\text{all space}}\int_{t_0}^{\infty} \rho(\vec{r}',t')\,G(\vec{r},\vec{r}',t-t')\,dt'\,d^3r'$$

or

$$\boxed{\phi_p(\bar{r},t) = \int_{\text{all space}} \frac{\rho\left(\bar{r}', t - \frac{|\bar{r}-\bar{r}'|}{v}\right)}{4\pi|\bar{r}-\bar{r}'|} \, d^3r'}$$